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FOUNDATIONS, THEORY OF SETS, LOGIC

Glenn, Oliver E. Mathematics and reality (a classic view). Math. Mag. **30** (1957), 117–126.

Miller, Hugh. Mathematics and reality (a modern view). Math. Mag. **30** (1957), 127–133.

Shepherdson, J. C. Note on a system of Myhill. J. Symb. Logic **21** (1956), 261–264.

This paper deals with two systems of Myhill, namely his system K [same J. **15** (1950), 185–196; MR **12**, 579] and an extension of K [ibid. **18** (1953), 7–10; MR **14**, 938], called K_1 in the present paper. The basic entities in K are finite sequences of non-negative integers (henceforth called sequences). Rational numbers are identified with certain sequences. Sets of rational numbers are therefore identified with sets of certain sequences. Myhill mentioned two ways of defining a real number x, namely, (1) by using its half-section, i.e., the set of all rationals r satisfying $r \leqq x$, and (2) by using its whole-section, i.e., the relation between rationals r and s which holds when $r \leqq x \leqq s$. In the two papers cited above Myhill briefly mentioned bounded classes of real numbers, i.e., bounded classes of half-sections (or of whole-sections).

In the present paper the classes of sets of sequences which are definable in K_1 are completely characterized. It follows from this characterization that no class consisting entirely of half-sections or of whole-sections is definable in K_1. The author concludes that K_1, though adequate for dealing with individual real numbers and operations on them, is not adequate for dealing with classes of real numbers (assuming that real numbers are defined as half-sections or whole-sections).

J. C. E. Dekker (Princeton, N.J.).

Kalmár, László. Ein direkter Beweis für die allgemein-rekursive Unlösbarkeit des Entscheidungs-problems des Prädikatenkalküls der ersten Stufe mit Identität. Z. Math. Logik Grundlagen Math. **2** (1956), 1–14.

The author notes that previous proofs of the theorem here presented [cf. Church, J. Symb. Logic **1** (1936), 40–41; and Turing, Proc. London Math. Soc. (2) **42** (1936), 230–265] exhibit a common property. Each constructs a countable collection of problems in a formal language outside of the predicate calculus and formal arithmetic which admits no solution by a general recursive procedure, and to which the Entscheidungsproblem can be reduced. The present proof is direct in the sense that it requires no use of a formal language other than the first order predicate calculus and formal arithmetic.

The proof of the theorem is accomplished by the construction of a formula A of the predicate calculus which depends on natural numbers n and systems of equations S which define functions ϕ of one argument, such that A is satisfiable if and only if the formula representing $\phi(n)=0$ is not derivable from S.

E. J. Cogan.

Rice, H. G. On completely recursively enumerable classes and their key arrays. J. Symb. Logic **21** (1956), 304–308.

This paper is concerned with non-negative integers (numbers), collections of numbers (sets) and collections of sets (classes). $\Phi(n, x)$ denotes a specific partial recursive function of n and x such that every partial recursive function of one variable occurs in the sequence $\Phi(0, x)$, $\Phi(1, x)$, $\cdots$. Notations: ω_n=the range of $\Phi(n, x)$, $\theta_A=\{n|\omega_n \epsilon A\}$, F=the class of all recursively enumerable (r.e.) sets, Q=the class of all finite sets,

$$L(\alpha)=\{\sigma|\alpha C\sigma \text{ and } \sigma \epsilon F\},$$

$L(A)=\sum L(\alpha)$, α ranging over A. The class A is called r.e. if $A=\{\omega_n|n \epsilon \omega_i\}$ for some i, and completely recursively enumerable (c.r.e.) if θ_A is r.e. In this paper the author continues his study of c.r.e. classes [Trans. Amer. Math. Soc. **74** (1953), 358–366; MR **14**, 713].

An explicit definition is given of an effective enumeration without repetitions $\{\alpha_n\}$ of Q such that (1) the cardinality of α_n is a recursive function of n, and (2) $\alpha_i C \alpha_k$ implies $i \leqq k$. The class K of finite sets is said to be canonically r.e. (canonically recursive) if $\{n|\alpha_n \epsilon K\}$ is r.e. (respectively, recursive). Every canonically r.e. subclass of Q is r.e.; the converse is shown to be false. K is called a key array of S if K is a canonically r.e. subclass of Q such that $S=L(K)$. It is known that a r.e. class is c.r.e. if and only if it has a key array [Myhill and Shepherdson, Z. Math. Logik Grundlagen Math. **1** (1955), 310–317; MR **17**, 1039]. Theorem: Let A be c.r.e.; then $F-A$ is r.e. if and only if A has a canonically recursive key array. The core of a class A is defined as the class of all finite sets in A which have no proper subset in A. It is proved that every c.r.e. class which has a canonically recursive key array also has a canonically recursive core. An example is given of a c.r.e. class whose core is not r.e.

J. C. E. Dekker (Princeton, N.J.).

Uspenskiĭ, V. A. On computable operations. Dokl. Akad. Nauk SSSR (N.S.) **103** (1955), 773–776. (Russian)

This is a very condensed and technical paper. It is concerned with formulation of a concept of recursive enumerability of one set relative to other sets which is parallel to definition due to Kleene [cf. Introduction to metamathematics, Van Nostrand, New York, 1952, Ch. XI–XII; MR **14**, 525] and Post [Bull. Amer. Math. Soc. **50** (1944), 284–316; MR **6**, 29] of relative recursiveness and relative decidability. The notion is defined in terms of that of a computable operation, converting certain sets $S_1, \cdots, S_p$ into a set R. The definitions are too complicated to be given here in full; but the main features are as follows. The author introduces the set $\mathfrak{H}$ which is the least set of expressions formed from | and parentheses such that a) the void expression and the expression consisting of a single | are in $\mathfrak{H}$, b) if A is in $\mathfrak{H}$, so is (A), and c) if A and B are in $\mathfrak{H}$ so is AB (the concatenation of A and B). [Such an $\mathfrak{H}$ was used by Hilbert, Verh. 3. Inter-

nat. Math.-Kongresses, Heidelberg, 1904, Teubner, Leipzig, 1905, pp. 174–185.] This $\mathfrak{H}$ can be regarded as containing all the positive integers and also all finite sequences of its own elements. A topology is introduced so that all the subsets of $\mathfrak{H}$, and also all the finite subsets, form a connected (bi)compact T_0-space. There is then defined a notion "partial transformation" from a direct power $\mathfrak{H}^k$ to $\mathfrak{H}$, and there is associated with each such transformation a "graph" which is a subset of $\mathfrak{H}$. The transformation is called computable just when the graph is enumerable. Given p enumerable subsets $M_1, \cdots, M_p$ of $\mathfrak{H}$, a p-place computable operation is a transformation leading from $S_1, \cdots, S_p$, where $S_i \subseteq M_i$, to a subset R of $\mathfrak{H}$, which transformation is a) continuous in the above topology, and b) such that it induces a computable partial transformation from a certain subset of $\mathfrak{H}^k$ to $\mathfrak{H}$. The theorems, which are stated without proof, cover certain properties of this notion, its equivalence in principle to certain notions introduced previously by Kolmogoroff and Post, and relations of relative enumerability to relative recursiveness and relative solvability.

H. B. Curry (University Park, Pa.).

Uspenskiĭ, V. A. Systems of denumerable sets and their enumeration. Dokl. Akad. Nauk SSSR (N.S.) **105** (1955), 1155–1158. (Russian)

This is an abstract, stating a series of complex definitions and about a dozen theorems without proof. The reviewer is unable to give here more than a rough idea of the general nature of its contents. Given a set M, a mapping onto (all of) M from a set of natural numbers is called an enumeration of M; and the set M considered in connection with an enumeration α is called an enumerated set and designated $M\alpha$. The author is concerned with the case where M is a "system" of recursively enumerable sets. The main purpose is to formulate, from as general a point of view as possible, without having recourse to an explicit Gödelization, definitions of notions related to the computability of such a numeration. Such a system forms a T_0-space in the topology of the paper reviewed above. The author formulates criteria for calculability and potential calculability of such a numeration, of "covering" and "completely" covering numerations, of complete enumerability and complete decidability of a subsystem P of an enumerated class $M\alpha$, of calculable transformation from enumerated class to another, and of openness and continuity of operations.

His theorems include assertions concerning the relation of these notions to the computable operations of his earlier paper. Some of his assertions generalize those of H. G. Rice [Trans. Amer. Math. Soc. **74** (1953), 358–366; MR **14**, 713]. *H. B. Curry* (University Park, Pa.).

Mal'cev, A. I. On representations of models. Dokl. Akad. Nauk SSSR (N.S.) **108** (1956), 27–29. (Russian)

The author exhibits alternative representations of models and relational systems in the sense of Tarski [Nederl. Akad. Wetensch. Proc. Ser. A. **57** (1954), 572–581, 582–588; **58**, 56–64; MR **16**, 554] and applies to them the following "basic local theorem" established by Mal'cev [Ivanovo Uč. Zap. Ped. Inst. Fiz.-Mat. Fak. **1** (1941), 3–9]: For the consistency of an infinite system of formulas of the restricted predicate calculus allowing identity and any set of symbols for individual objects and predicates, it is necessary and sufficient that every finite subsystem of the system be consistent. This technique yields theorems which have as corollaries certain well known results on representations of algebras.

The final representation of models is carried out in a predicate calculus of higher order and yields the following theorem. Every partially ordered locally nilpotent group without cycles has a central system which consists of convex normal divisors. The result for groups with a finite number of generators is due to B. H. Neumann [Proc. London Math. Soc. (3) **4** (1954), 138–153; MR **17**, 448].

It can be shown directly from the definition of "proof" that a theorem analogous to the 1941 result cited above holds for any logical system. *E. J. Cogan.*

McNaughton, Robert. Logical and combinatorial problems in computer design. Computers and Automation **6** (1957), no. 1, part 1, 30–31.

See also: Korobkov, p. 372; Tiberti, p. 373; Routledge, p. 422.

ALGEBRA

Combinatorial Analysis

Hoffman, A. J.; and Kuhn, H. W. Systems of distinct representatives and linear programming. Amer. Math. Monthly **63** (1956), 455–460.

Let $F=\{S_1, \cdots, S_n\}$ be a family of subsets of some set S. A subset $R=\{a_1, \cdots, a_n\} \subset S$, of n distinct elements is called a system of distinct representatives (SDR) of F if $a_j \in S_j$, for $j=1, \cdots, n$. A classical theorem of P. Hall states: A necessary and sufficient condition that F admit a SDR is that (A) the union of every q of the sets S_j contains at least q elements.

Mann and Ryser [same Monthly **60** (1953), 397–401; MR **14**, 1053] have considered the following generalization of this problem. Given a set $E \subset S$, when is it possible to find a set R such that R is a SDR of F and $E \subset R$? A sufficient condition for the existence of such a set was given. In the present paper the authors replace the sufficient (but not necessary) condition of Mann-Ryser by the following weaker condition: (C) Every set $E' \subset E$ of p elements meets at least p distinct sets S_j.

It is then shown that (A) and (C) together are necessary and sufficient for the generalized problem to have a solution. The theorem is proved by means of the theory of linear inequalities, specifically, the duality theorem of linear programming and a property of solutions to transportation problems. *D. Gale.*

McClenon, R. B. Pascal's arithmetical triangle. Proc. Iowa Acad. Sci. **63** (1956), 534–537.

See also: Hoffman and Kuhn, p. 416.

Linear Algebra

Jaeckel, K. Über eine Matrizentransformation mit Dreiecksmatrizen. Z. Angew. Math. Mech. **36** (1956), 154–155.

The author proves the following form of the triangular decomposition theorem, valid for an arbitrary square

matrix $\mathfrak{A}$ of complex numbers: There exists an upper triangular matrix $\mathfrak{O}$ and a lower triangular matrix $\mathfrak{U}$ such that $\mathfrak{A}\mathfrak{O}=\mathfrak{U}$. The elements of $\mathfrak{O}$ and $\mathfrak{U}$ are represented in terms of subdeterminants of $\mathfrak{A}$. If $\mathfrak{A}$ is Hermitian, it follows that $\bar{\mathfrak{O}}'\mathfrak{A}\mathfrak{O}=\mathfrak{D}$, where $\mathfrak{D}$ is a real diagonal matrix. The author then quickly derives the known theorem that a Hermitian matrix $\mathfrak{A}$ is positive definite when all its principal minors are positive.

G. E. Forsythe (Los Angeles, Calif.).

Afriat, S. N. On the latent vectors and characteristic values of products of pairs of symmetric idempotents. Quart. J. Math. Oxford Ser. (2) **7** (1956), 76–78.

Let e, f be [square] symmetric idempotent matrices, of the same dimension. Then there exist matrices E, F such that $e=EE^*$, $f=FF^*$, $E^*E=I$, $F^*F=I$,

$$E^*F=\operatorname{diag}(\rho_1, \rho_2, \cdots, \rho_r, 0, \cdots, 0).$$

Theorems concerning the characteristic vectors of e, f are given, and it is shown that if $\lambda=x^* ef x$, then $|\lambda|\leqq 1$ and $\lambda=\bar{\lambda}^*$.

J. L. Brenner (Palo Alto, Calif.).

Černikov, S. N. On strictly nonvanishing solutions of a system of linear equations. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 2(68), 223–228. (Russian)

Simple connection [incomplete in Gummer, Amer. Math. Monthly **33** (1926), 488; erroneous in Mihel'son, Uspehi Mat. Nauk (N.S.) **9** (1954), no. 3(61), 163–170; MR **16**, 105] between the existence of a positive solution of a system of linear equations and that of certain non-negative solutions.

T. S. Motzkin.

Remez, E. Ya. Questions of uniqueness or multiplicity of solutions of the Čebyšev problem for a system of incompatible linear equations and the concept of a normal Čebyšev solution. Ukrain. Mat. Ž. **8** (1956), 34–53. (Russian)

Whereas the solutions of the minimax error problem $\min_x \max_i |\varepsilon_i|$, $\varepsilon_i=\sum_{k=1}^{n} a_{ik}x_k+b_i$ with rank $(a_{ik})=n$ form a convex polyhedron, only one, the "normal" solution satisfies the strong extremization requirement $\min_x{}^* \varepsilon$: that the set of $|\varepsilon_i|$, arranged in decreasing order, precede lexicographically, or coincide with, the corresponding set for every other x. Also a numerical example, history, and literature. {Reviewer's remark: the same idea is applicable to best solutions of (not necessarily incompatible) inequalities: among them, the normal solutions always form a, possibly empty, linear manifold.}

T. S. Motzkin (Los Angeles, Calif.).

Braumann, Pedro. A theorem about systems of linear equations. Univ. Lisboa. Revista Fac. Ci. A. (2) **5** (1955–1956), 103–112.

If $a_{ik}>0$ ($i=1, \cdots, n$; $k=1, \cdots, n+1$) and $a_{ii}a_{i+1\,k}<$ or $>a_{i+1\,i}a_{ik}$ as $k<i$ or $k>i+1$, then the determinant and solution of $\sum \varepsilon_{ik}a_{ik}z_k=a_{i\,n+1}$, $\varepsilon_{ik}=1$ or -1 as $k\leqq i$ or $k>i$, are positive; and similar reults.

T. S. Motzkin.

Rado, R. Note on generalized inverses of matrices. Proc. Cambridge Philos. Soc. **52** (1956), 600–601.

It is shown that the generalized inverse of an $m\times n$ matrix introduced by Penrose [same Proc. **51** (1955), 406–413; MR **16**, 1082] had already been introduced in a form only slightly different by E. H. Moore [General analysis, v. 1, Amer. Philos. Soc., Philadelphia, 1935]. An alternative treatment for Moore's result is presented valid for matrices in any division ring with an involutory anti-automorphism $x\to\bar{x}$ such that $\sum x_i\bar{x}_i=0$ implies $x_i=0$.

O. Taussky-Todd (Washington, D. C.).

See also: Zuhovickiĭ, p. 391; Varga, p. 414; Černikov, p. 417; Burger, p. 417; Kuhn, p. 417; Young, p. 417; Kosko, p. 418; von Holdt, p. 418; Prager, p. 451; Kavanagh, p. 453.

Polynomials

Klingenberg, Wilhelm. Die Anzahl der Nullstellen eines Polynoms in Gebieten mit stückweise rationalen Randkurven. Z. Angew. Math. Phys. **7** (1956), 304–316.

The author determines the number Z of zeros of a polynomial $p(z)$ in a region bounded by a closed rectifiable curve c which is mapped by $w=p(z)$ upon a closed curve $C=\sum_{k=1}^{r} C_k$, where each arc C_k has an equation of the form

$$w_k(t)=\{P_k(t)+iQ_k(t)\}/R_k(t) \quad (a_k\leqq t\leqq b_k),$$

and P_k, Q_k, R_k are real polynomials with

$$\{P_k(t)^2+Q_k(t)^2\}R_k(t)\neq 0 \quad (a_k\leqq t\leqq b_k).$$

Let $W_k(t)$ be the number of variations of sign in the Sturm sequence which begins with $P_k(t)$, $-Q_k(t)$ when the degree of P_k is not less than that of Q_k, and which begins with $Q_k(t)$, $P_k(t)$ when the degree of P_k is not greater than that of Q_k. Let $N_k(t)$ be defined as zero when $Q_k(t)=0$ or $P_k(t)Q_k(t)<0$ and as one when $P_k(t)=0$ or $P_k(t)Q_k(t)>0$. Let finally

$$Z_k=\tfrac{1}{2}\{W_k(a_k)-W_k(b_k)\}+\tfrac{1}{2}\{N_k(a_k)-N_k(b_k)\}.$$

Then, by using reasoning familiar with Sturm sequences, the author establishes that $Z=\sum_{k=1}^{r} Z_k$.

M. Marden.

Mitrović, Dusan. Conditions graphiques pour que l'argument de chacune des racines d'une équation algébrique soit compris entre $(\pi/2)+\mu$ et $(3\pi/2)-\mu$. C. R. Acad. Sci. Paris **243** (1956), 831–833.

Given the polynomial $f(z)=a_0+a_1z+\cdots+a_nz^n$ in which all the a_k are real and $a_n>0$. Let $\phi_j(t)$ be defined by the recursion

$$\phi_j(t)=-2t\phi_{j-1}(t)-\phi_{j-2}(t) \quad (j=2, 3, \cdots, n),$$

with $\phi_0(t)=0$, $\phi_1(t)=1$. With $t=-\sin\mu$, and $0\leqq\mu\leqq\pi/2$, let K be the curve described by the point (ξ, η) defined by

$$\xi=\sum_{j=2}^{n} a_j\phi_j(t)r^{j-1}, \quad \eta=-r^2\sum_{j=2}^{n} a_j\phi_{j-1}(t)r^{j-2}$$

when r varies from 0 to ∞. Then all the zeros of $f(z)$ lie in the sector $(\pi/2)+\mu\leqq \arg z\leqq(3\pi/2)-\mu$ if the point (a_1, a_0) lies in the first quadrant and if, as r varies from 0 to ∞, the curve K intersect the lines $\eta=a_0$ and $\xi=a_1$ alternately starting with line $\eta=a_0$ and if the total number of intersections is m, the largest positive integer which satisfies the inequality

$$(\pi/2)m+\{1-(-1)^m\}(\mu/2)<\{(\pi/2)+\mu\}n.$$

The proof consists in showing that as r varies from $-\infty$ to $+\infty$, $\arg f(ire^{i\mu})$ varies by the amount $n\pi$. When $\mu=0$, the result reduces to that given earlier [same C. R. **240** (1955), 1177–1179; MR **16**, 1019].

M. Marden.

★ **Abel, N. H. Mémoire sur les équations algébriques où on démontre l'impossibilité de la résolution de l'équation générale du cinquième dégré.** Facsimile edition. The Librarian, Faculty of Science, University of Oslo, Oslo, 1957. [Originally published: Christiania. De l'imprimerie de Groendahl, 1824.] 8 pp.

Korobkov, V. K. Realization of symmetric functions in the class of π-circuits. Dokl. Akad. Nauk SSSR (N.S.) **109** (1956), 260–263. (Russian)

For the study of π-circuits any symmetric Boolean function with n variables $f(x_1, \cdots, x_n)$ is written with the basic operations &, $\vee$ and $^-$ as an r-fold disjunction of n-fold conjunctions each of whose factors is either x_j or $\bar{x}_j$ $(j=1, \cdots, n)$. For each $i=1, \cdots, r$, let k_i (the working number) be the number of negations in the i-th disjunct. If $r=1$, f is an elementary symmetric function and we let $k_1=k$. Let S_n^k be the number of occurrences of the variables $x_1, \cdots, x_n$ in an elementary symmetric function f of n variables whose working number is k. It is shown that for $1\leq k\leq n$, $S_n^k<18.75\cdot 2^p$, where

$$p=\tfrac{1}{2}((\log_2 n)^2+\log_2 n).$$

An upper bound is also furnished for numbers of occurrences of variables in arbitrary symmetric functions of n variables represented in this normal form, and a table of values for S_n^k is provided for $1\leq k\leq n$ and $1\leq n\leq 8$.

E. J. Cogan (Hanover, N.H.).

See also: Carlitz, p. 377; Parodi, p. 385; Sispánov, p. 420.

Theory of Invariants

Larionov, B. A. On the rational basis of covariants of an n-ary form. Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. **16** (1955), 87–105. (Russian)

The author considers the mutually inverse linear transformations

$$x_i=\sum p_{ki}z_k,\quad z_k=\sum' q_{ik}x_i$$

and introduces quantities, functions of x of degree p, indicated by

$$\overset{p}{a}{}^{p_1\cdots p_n}_{q_1\cdots q_n},$$

which per definition satisfy certain rules for differentiation. He considers the effect of operations

$$d_{ks}=\sum_i q_{si}\frac{\partial}{\partial q_{ki}},\quad \bar{d}_{ks}=\sum_i p_{si}\frac{\partial}{\partial p_{ki}}$$

and introduces the concept incomplete invariant B of the first kind, which differs from the semi-invariant by requiring next to

$$d_{ks}B=0,\ k\neq s,\ k\neq 1,$$

also

$$d_{kk}B=q_kB,\ k=s.$$

Semi-invariants do not satisfy the second set of relations and invariants are obtained if the first set is also true for $k=1$. He introduces also the concept of incomplete invariants of the second kind b, by requiring

$$\bar{d}_{ks}b=0,\ k\neq s,\ s\neq 1,$$
$$\bar{d}_{kk}b=q_kb,\ k=s.$$

The author deduces a system of exactly one hundred formulas for these quantities and states a one-to-one correspondence between the incomplete invariants of an n-ary form of degree m and the simultaneous invariants of m forms of degree $1, 2, \cdots, m$ in the $(n-1)$-ary domain, from which several theorems concerning the minimal complete system are derived.

{If one introduces the fundamental points and hyperspaces ξ_i, resp. ξ_i',

$$\xi_1=(1, 0, \cdots, 0), \cdots, \xi_n'=(0,0,\cdots,1),$$

one obtains introducing the points

$$P_k=(p_{k1}, p_{k2}, \cdots, p_{kn}),\ Q_k=(q_{k1}, q_{k2}, \cdots, q_{kn})$$

that $(a'\xi)=(\bar{a}'Q)$ and $(\bar{a}'\bar{\xi})=(a'P)$, where the bar denotes the transformed coefficients of $(a'x)^m$. We then have

$$\overset{p}{a}{}^{p_1\cdots p_n}_{q_1\cdots q_n}=(a'x)^p(a'P_1)^{p_1}\cdots(a'P_n)^{p_n}(a'\xi_1)^{q_1}\cdots(a'\xi_n)^{q_n}$$

and the correspondence of the polar operations

$$d_{ks}=D_{Q_kQ_s}\sim D_{\xi_k\xi_s},\ \bar{d}_{ks}=D_{P_kP_s},\ \frac{\partial}{\partial z_k}=D_{xP_k}$$

$$D_{\xi_k\xi_s}(x\xi_k')=D_{\xi_k\xi_s}(x\xi_1\cdots\xi_{k-1}\xi_{k+1}\cdots\xi_k)=-(x\xi_s')$$

reduces the greater part of the relations derived by the author to trivialities, whereas the identity

$$\overset{p}{a}{}^{p_1\cdots p_n}_{q_1\cdots q_n}=\overset{p}{\bar{a}}{}^{q_1\cdots q_n}_{p_1\cdots p_n}$$

shows, that the distinction between incomplete invariants of the first and second kind is irrelevant [see Weitzenböck, Invariantentheorie, Noordhoff, Groningen, 1923, p. 140, formulas (4), (7)]. Applying the result contained in Weitzenböck, Invariantentheorie, p. 141, between formulas (10) and (11) we have obviously

$$B(a_{q_1\cdots q_n})=E^q(\alpha'\xi_1)^p,$$

where

$$E=\det\sum\pm(\varphi_1'\xi_1)\cdots(\varphi_n'\xi_n)=(\varphi_1'\varphi_2'\cdots\varphi_n')(\xi_1\xi_2\cdots\xi_n)$$

with

$$(\varphi_k'x)^q=(a_1'x)^{\alpha_{k1}}(a_2'x)^{\alpha_{k2}}\cdots(a_s'x)^{\alpha_{ks}},$$

$\alpha_{k1}+\alpha_{k2}+\cdots+\alpha_{ks}=q$ and therefore

$$B(a_{q_1\cdots q_n})=(\varphi_1'\varphi_2'\cdots\varphi_n')^q(\xi_1\xi_2\cdots\xi_n)^q(a'\xi_1)^{p_1}\cdots(a'\xi_n)^{p_n},$$
$$p_1+p_2+\cdots+p_n=p.$$

Obviously every incomplete invariant corresponds to some simultaneous invariant of the n-ary forms of degree $1, 2, 3, \cdots, m$

$$(a_1'x), (a_2'x)^2, (a_3'x)^3, \cdots, (a_s'x)^m$$

obtained from B by leaving out the factors $(a'\xi_1)$ and the determinant factors and only considering the function of $a_1', \cdots a_s'$, corresponding to $(\varphi_1'\varphi_2'\cdots\varphi_n')^q$.

The theorem stated by the author concerning a one-to-one correspondence between the incomplete invariants of an n-ary form and simultaneous invariants of $(n-1)$-ary forms must therefore be wrong.

One of the drawbacks of the kernel-index-notation is that the splitting up in determinant factors is not so simple. The Clebsch-Weitzenböck notation gives simply $B(a^{0q_2\cdots q_n}_{q_10\cdots 0})$

$$=(\varphi_1'\varphi_2'\cdots\varphi_n')^q(\xi_1P_2P_3\cdots P_n)^q(a_1'\xi_1)^{p_1}\cdots(a_s'\xi_1)^{p_s}$$

which puts into evidence that these functions are sym-

bolical invariants under linear transformation of $P_1, P_2, \cdots, P_n$. As these $(n-1)$-ary invariants contain symbolical coefficients, to an incomplete invariant of an n-ary form there does not correspond one simultaneous invariant of $(n-1)$-ary forms but a set of simultaneous invariants

$$(\varphi_2'\varphi_3'\cdots\varphi_n')^q, (\varphi_1'\varphi_3'\cdots\varphi_n')^q, \cdots$$
$$\cdots, (\varphi_1'\varphi_2'\cdots\varphi_n')^q, \cdots, (\varphi_1'\varphi_2'\cdots\varphi_{n-1}')^q.$$

Using the kernel-index-notation as the author does, the obvious relation

$$B(a^{0q_2\cdots q_n}_{q_10\cdots 0})=(P_2P_3\cdots P_n)^q B(a_{q_1\cdots q_n}),$$

requires rather clumsy analytical operations; it screens the fact that many of the relations obtained concern only the determinant $(\xi_1P_2\cdots P_n)$ and in fact are simple, well-known relations between their minors requiring only algebraical deductions. The screening of the symbolical coefficients in B causes the author to state a theorem that can not be true.} *E. M. Bruins* (Amsterdam).

Continued Fractions

See: Rutishauser, p. 418.

Partial Order Structures

Padmavally, K. On a problem of G. Kurepa. Proc. Nat. Inst. Sci. India. Part A. **21** (1955), 368–372 (1956).

The problem referred to is: Does there exist an ordered set S such that for every maximal antichain A of S, each maximal chain of $S-A$ is also one in S? The reciprocal question obtained from the latter by permuting all over, chain–antichain, is relatively easy to solve: such is every ramified set in which each point has at least 2 incomparable (not necessarily immediate) successors. Now, the first problem in the paper is reduced to the second one by reordering reciprocally each ramified table T; this means that 2 points are comparable in one ordering if and only if the same points are incomparable in the other ordering. *Đ. Kurepa* (Zagreb).

McLaughlin, J. E. Atomic lattices with unique comparable complements. Proc. Amer. Math. Soc. **7** (1956), 864–866.

A theorem of the reviewer to the effect that a finite dimensional complemented lattice with unique comparable complements is modular [Tôhoku Math. J. **47** (1940), 18–23; MR **2**, 120] is shown to be valid if the condition of finite dimensionality is replaced by atomicity. This is a very considerable extension of the range of validity of the theorem and requires an entirely new technique of proof. It follows that every atomic lattice with unique complements is a Boolean algebra. This improves the corresponding theorem of Birkhoff-Ward by removing the condition of completeness. *R. P. Dilworth.*

Tiberti, Maria Rosaria. Universo di Boole, di Reichenbach e misti. Giorn. Mat. Battaglini (5) **4**(84) (1956), 5–17.

A derivation of some trivial identities in Boolean algebras and Post algebras [cf. Rosenbloom, The elements of mathematical logic, Dover, New York, 1950; MR **12**, 789]. The latter are here called Reichenbach algebras.
P. R. Halmos (Chicago, Ill.).

See also: Korobkov, p. 372; Robinson, p. 374; Plotkin, p. 448.

Rings, Fields, Algebras

★**Jacobson, Nathan. Structure of rings.** American Mathematical Society, Colloquium Publications, vol. 37. American Mathematical Society, 190 Hope Street, Prov., R. I., 1956. vii+263 pp.

The subject of this book is the structure of (noncommutative) rings without finiteness assumptions. Among other things, the majority of the results in the author's "Theory of rings" [Amer. Math. Soc. Math. Surveys, v. I, New York, 1943; MR **5**, 31] and in Artin, Nesbitt, and Thrall's "Rings with minimum condition" [Univ. of Michigan Press, 1944; MR **6**, 33] are generalized, in particular those on semi-simple rings. Certain generalizations of older concepts have played a key role in the emancipation of the structure theory of rings from chain conditions. The author's exploitation of Perlis' characterization of the radical of a finite dimensional algebra, the replacement of simple rings by primitive rings, and the use of modular (=regular) maximal right ideals instead of idempotents and minimal right ideals are among those essential developments in this book. There have been a large number of contributions made to the structure theory of rings without finiteness assumptions, especially in the last fifteen years. The author concerns himself primarily (but by no means exclusively) with those of Amitsur, Azumaya, Baer, Chevalley, Dieudonné, Kaplansky, Kurosch, Levitzki, McCoy, Nakayama, and himself. We give next an (incomplete) summary of the contents of the book.

Chapter I is concerned primarily with the (Perlis-Jacobson) racidal and semi-simplicity, but along the way some of the most important concepts in the book are introduced, such as irreducible module, primitive ring and primitive ideal, modular (=regular) ideal, and subdirect sum. In addition, most of the notational conventions peculiar to this book are introduced at this point. Chapter II is about irreducible modules and primitive rings. The central theme of the chapter is a density theorem on irreducible modules (which is perhaps more familiar as a theorem on rings of linear transformations). It is formulated both in a purely algebraic and in a topologico-algebraic way, and is used frequently in the sequel. In Chapter III the classical structure theorems for rings satisfying the minimum condition for right ideals are derived in short order with the aid of the previous material. Next the theory of idempotent elements and matrix units in arbitrary rings is developed for immediate and subsequent use. Chapter IV is devoted to the study of primitive rings having minimal one-sided ideals. By strongly exploiting duality in vector spaces, it is shown that if such a ring has a non-zero socle (=the sum of the irreducible right ideals), then it is representable as a ring of continuous linear transformations. The special case of simple rings having minimal one-sided ideals is studied in some detail. Chapter V is devoted to Kronecker products of rings and modules; in particular to the study of Kronecker products of algebras of known structure (simple algebras, primitive algebras, and radical algebras, among others). In some cases, a reduction to division rings is accomplished, and the matter is taken up again in Chapter VII. In Chapter VI, completely reducible modules and the Galois theory of the ring of all linear

transformations of a vector space are studied. In Chapter VII, division rings are considered in some detail. First of all, some topics considered earlier and reduced to the division ring case are taken up again here (e.g., Kronecker products). Secondly, some topics developed previously in a more general context are redeveloped here, because the more restrictive hypothesis makes simplification possible (e.g., Galois theory). In addition, the structure of division rings is considered (e.g., the Wedderburn theorem that every finite division ring is a field, and the Cartan-Brauer-Hua theorem). Chapter VIII is concerned with nil ideals and prime ideals. Various nil radicals are considered; in particular the lower nil radical of Bear (which is shown to coincide with the intersection of all the prime ideals), and Levitzki's nil radical (=the sum of all the locally nilpotent ideals) are studied. In Chapter IX the basic properties of the structure space of an arbitrary ring (i.e., the space of primitive ideals of the ring topologized by letting the closure of a set A of primitive ideals be the set of primitive ideals containing the intersection of all the elements of A) are obtained. If a (semi-simple) ring is a subdirect sum of primitive rings, and if all the summands are isomorphic, then the given ring is a ring of functions on its structure space. Under special circumstances, these functions are continuous. For certain bi-regular rings this turns out to be the case, and these are represented as the ring of all continuous functions with compact carriers on locally compact totally disconnected spaces to simple (discrete) rings with unit (Arens-Kaplasnky). In Chapter X, the book closes with several applications of the structure theory. Its highlights are certain commutativity theorems (e.g., if for every x in a ring A, there is an integer $n(x)>1$ such that $x^{n(x)}-x$ is in the center of A, then A is commutative (Herstein)), algebras satisfying a polynomial identity (PI-algebras), Kaplansky's solution of Kurosch's problem for PI-algebras, and some recent work of Amitsur on algebraic algebras.

A bibliography, primarily of the literature since 1943, is given. (The reader is referred to the author's "Theory of rings" for earlier references.) Its value would have been increased if the author had written some historical notes and some brief commentary on relevant papers whose contents are not included in the text. In his preface the author states that he had planned to write such notes, but refrained from doing so to facilitate earlier publication.

{The book seems to be addressed primarily to experts in the field or to mathematicians who have a substantial knowledge of the theorems and techniques of modern algebra. The author's statement in the preface "The only knowledge assumed is that of the rudiments of ring and module theory such as is found in any of the introductory texts to abstract algebra." is, at a minimum, misleading. Far too many "well-known" theorems are quoted without reference to make it possible for a student to read it without substantial guidance from a more experienced person. (At one point in the proof of Theorem 9.2.3, the reader is expected to use (without further explanation or reference), "the usual Vandermonde determinant argument." But even the term "Vandermonde determinant" does not appear either in the index of van der Waerden's "Moderne Algebra" [2 vols., 2nd ed., Springer, Berlin, 1937, 1940; MR **2**, 120] or in the index of either of the two volumes of the author's "Lectures in abstract algebra" [Van Nostrand, New York, 1951, 1953; MR **12**, 794; **14**, 837] that have appeared so far, or in the present book.)

The exposition is complete, but terse. Almost always, the author proceeds from the general to the particular. Motivation is usually given, but often is missing. On the positive side, the material of Chapters VIII, IX, and X is especially well-motivated. On the other hand, the reader unfamiliar with Kronecker products has to wait a while before discovering why they are studied. In the same vein, semisimple and primitive rings are introduced on page 4, but simple rings are not introduced until page 39 (where it is shown that in the presence of the minimum condition for right ideals, a ring is primitive if and only if it is simple). Not until page 93 will the reader find a reasonably explicit example of a primitive ring that is not simple (although he will suspect its existence before then). Also, some unnecessary confusion is introduced by using the same symbol for a function and the value of the function at an arbitrary but fixed point — in particular in discussing subdirect sums of rings in Chapter I.

The reviewer discovered no errors, but there are a goodly number of trivial misprints. One which may delay the reader a bit occurs in Proposition 5.1.2. In line 2, change "$\mathfrak{M}_1$" to "$\mathfrak{M}_1'$" and change "$\mathfrak{L}_1$" to "$\mathfrak{L}$".

The above criticisms notwithstanding, the author has done an impressive job of gathering together the bulk of the important contributions to the structure theory of rings without finiteness assumptions, a service for which all present and future research workers in this area should be thankful. It is replete with interesting unsolved problems, and will undoubtedly become a standard reference for some years to come. It represents a valuable addition to the Colloquium series.} *M. Henriksen.*

Robinson, Abraham. Solution of a problem by Erdös-Gillman-Henriksen. Proc. Amer. Math. Soc. **7** (1956), 908–909.

It is well-known (and easily seen) that (a) any isomorphism of a real-closed field is necessarily order-preserving. But the converse does not hold; in fact, (b) let t_1, t_2 be independent real transcendentals over the rational field R, and let R_k denote the real-closure of $R(t_k)$; then R_1, R_2 are similarly ordered, but not isomorphic. The authors mentioned in the title proved a partial converse of (a), namely, (c) for each $\alpha>0$, all real-closed η_α-fields of power $\aleph_\alpha$ are isomorphic (an ordered field F is an η_α-field if between any two subsets A, B of power $<\aleph_\alpha$, with $A<B$, there lies an element of F), and asked the following question: (d) more generally, are all similarly ordered nondenumerable nonarchimedean real-closed fields isomorphic? [[1] Ann. of Math. (2) **61** (1955), 542–554; [2] MR **16**, 993]. In the paper under review, the author constructs a simple example to show that the answer to (d) is negative. In the notation of (b), let R_k^* denote the field of fractional power series

$$a=\sum_{j=-m}^{\infty} a_j x^{j/n} \quad (a_j \in R_k;\ n=1, 2, \cdots;\ m=0, 1, \cdots)$$

($a>0$ if and only if $a_j>0$ for the nonvanishing a_j with lowest subscript); then R_k^* is real-closed, nonarchimedean and of power c; and R_1^*, R_2^* are similarly ordered, but not isomorphic.

There are a number of misstatements in the paper, in [1], and in [2] (some pointed out by Henriksen); while they do not invalidate the results, they can easily cause confusion. The present paper refers to real-closed fields related by an isomorphism that does not preserve order; the reference should be instead to similarly ordered fields that are not isomorphic. It also states that R_1 is isomor-

phic with R_2, and that R_1^* is isomorphic with R_2^*; these statements should be deleted. As for the other papers, in [1], the conclusion of (b) is given in terms of $t_1=e$, $t_2=\pi$, thereby (as pointed out by W. J. LeVeque) asserting their algebraic independence, which is still an open question. In [2], it is stated that an algebraically closed field of characteristic 0 is determined by its cardinal number; replace "cardinal number" by "transcendence degree over the rationals". Finally, the quotation of (c) in [2] omits the phrase "of power $\aleph_a$"; in the absence of this hypothesis, the problem is still unsettled.

L. Gillman (Lafayette, Ind.).

Leavitt, William G. Two word rings. Proc. Amer. Math. Soc. 7 (1956), 867–870.

The author gave several conditions in an earlier paper [An. Acad. Brasil. Ci. **27** (1955), 241–250; MR **17**, 578] that a ring be dimensional. One of these conditions is weakened in the present paper, and a word ring is constructed satisfying this new condition but none of the previous ones. Another word ring is constructed to show that a ring K need not be imbeddable in a ring with descending chain condition on right ideals in order that all finitely based modules over K have an invariant basis number. *R. E. Johnson* (Northampton, Mass.).

Kohls, Carl W. On the embedding of a generalized regular ring in a ring with identity. Michigan Math. J. 3 (1955–1956), 165–168.

Let A denote a ring, n the characteristic of A, and I_n the ring of integers modulo n. If

$$(A;I_n)=\{(a,s): a\in A, s\in I_n\},$$

where

$$(a,s)+(b,t)=(a+b,s+t),\quad (a,s)(b,t)=(ab+sb+at,st),$$

then (as is well-known) $(A;I_n)$ is a ring of characteristic n with identity element (0, 1) that contains A as an ideal.

The main result of the paper is the following. If A is commutative, then $(A;I_n)$ is regular (in the sense of von Neumann) if and only if A is regular and $n\neq 0$. {Reviewer's remark: A corresponding theorem is obtained for commutative semi-simple π-regular rings, but since every such ring is regular, [cf. e.g., Kaplansky, Portugal Math. **10** (1951), 37–50; MR **13**, 8], the added generality is illusory.} *M. Henriksen* (Princeton, N.J.).

Villamayor, Orlando E. On the theory of unilateral equations in associative rings. Rev. Mat. Cuyana 1 (1955), 1–40.

This paper is essentially the presentation of the proofs, in detail, of theorems announced previously [C. R. Acad. Sci. Paris **240** (1955), 1681–1683, 1750–1751; MR **17**, 121].

I. N. Herstein (New Haven, Conn.).

Harada, Manabu. Note on the dimension of modules and algebras. J. Inst. Polytech. Osaka City Univ. Ser. A. 7 (1956), 17–27.

Let Λ be a ring with unit. Following the definitions given in "Homological algebra" [Princeton, 1956; MR **17**, 1040] by H. Cartan and S. Eilenberg, the author defines the left global dimension of Λ (notation: l.gl.dim Λ) to be properly less than n if the functor $\mathrm{Ext}_\Lambda{}^n=0$. Similarly the weak global dimension of Λ (notation: w.gl.dim Λ) is defined to be properly less than n if the functor $\mathrm{Tor}_n{}^\Lambda=0$.

The following results are established. If Λ_n is the full $n\times n$ matrix ring over Λ, then l.gl.dim $\Lambda=$l.gl.dim Λ_n and w.gl.dim $\Lambda=$w.gl.dim Λ_n. If Λ is an algebra over a commutative ring K, then it is shown that dim $\Lambda=$ dim Λ_n, where dim Λ denotes the algebra dimension of Λ. It is also shown that w.gl.dim $\Lambda=$sup w.gl.dim Λ/I, where I runs through all left ideals in Λ. The following characterization of a ring being regular in the sense of von Neumann [Proc. Nat. Acad. Sci. U.S.A. **22** (1936), 707–713] is established, namely Λ is regular if and only if w.gl.dim $\Lambda=0$. Some results on "change of rings" are obtained. In particular, it is shown that if I is a two sided ideal equal to Λe or $e\Lambda$ where e is idempotent, then l.gl.dim $\Lambda\geqq$l.gl.dim Λ/I and w.gl.dim $\Lambda\geqq$w.gl.dim Λ/I.

M. Auslander (Princeton, N.J.).

Jenner, W. E. On the class number of non-maximal orders in $\mathfrak{p}$-adic division algebras. Math. Scand. 4 (1956), 125–128.

In analogy to the case of rational algebras [Zassenhaus, Abh. Math. Sem. Hansischen Univ. **12** (1938), 276–288], the paper shows that the left and right class numbers of an order of a semisimple algebra over the p-adic numbers are equal and finite. A corollary is that the number of unimodularly inequivalent integral representations, with a given degree, of a finite group in a $\mathfrak{p}$-adic number field is finite. *T. Nakayama* (Nagoya).

Albert, A. A. A property of special Jordan algebras. Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 624–625.

Answering a question posed by N. Jacobson, the author shows that the exceptional Jordan algebra is not a homomorphic image of a finite-dimensional special Jordan algebra. It remains possible that it is such an image of an infinite-dimensional special Jordan algebra, or even that any free Jordan algebra is special. *I. Kaplansky.*

Patterson, E. M. On the generators of nilpotent linear algebras. Quart. J. Math. Oxford Ser. (2) 7 (1956), 17–23.

Let A be a nonassociative algebra of dimension n. Define right powers A^i (respectively left powers A_i, derived algebras $A^{(i)}$) by $A^1=A$, $A^{i+1}=A^iA$ ($A_1=A$, $A_{i+1}=A_iA$; $A^{(0)}=A$, $A^{(i+1)}=A^{(i)}A^{(i)}$). Denote by n_i the dimension of $A^{(i)}$. If N is the minimum number of generators of A, call the difference $g=n-N$ the genus of A. Assume that A satisfies condition (α): any two subspaces A^j, A_k satisfy either $A^j\subseteq A_k$ or $A_k\subseteq A^j$ (for example, (α) is satisfied in associative, commutative, or anticommutative algebras since $A^k=A_k$). Theorem: If a nilpotent algebra A satisfies (α), and if $e_1,\cdots,e_n$ is a basis for A such that $e_1,\cdots,e_{n_1}$ is a basis for $A^{(1)}$, then $g=n_1$ and A is generated by $e_{n_1+1},\cdots,e_n$. Corollary: If A is a solvable Lie algebra of characteristic 0, then $n_2\leqq g\leqq n_1$. *R. D. Schafer* (Storrs, Conn.).

Kokoris, Louis A. Simple power-associative algebras of degree two. Ann. of Math. (2) **64** (1956), 544–550.

Examples are known of simple commutative power-associative algebras of degree two and characteristic $p>5$ which are not Jordan algebras [Kokoris, Proc. Nat. Acad. Sci. U.S.A. **38** (1952), 534–537; MR **14**, 11; Albert, Trans. Amer. Math. Soc. **74** (1953), 323–343; MR **14**, 614]. In this paper the author proves that all simple commutative power-associative algebras of degree two and characteristic 0 are Jordan algebras. This, coupled with known results of Albert [ibid. **69** (1950), 503–527; MR **12**, 475],

yields the theorem: Every semisimple commutative power-associative algebra of characteristic 0 is a Jordan algebra. *R. D. Schafer* (Storrs, Conn.).

See also: Higman, p. 377; Carlitz, p. 377; Pyateckiĭ-Šapiro, p. 378; Warmus, p. 391; Weiss, p. 406; Postnikov, p. 409; Engel, p. 415.

Groups, Generalized Groups

Karrass, A.; and Solitar, D. Some remarks on the infinite symmetric groups. Math. Z. **66** (1956), 64–69.

A number of facts about infinite symmetric groups (normalisers of elements, orders of composition factors, etc.,) together with the observation that a non-trivial normal subgroup of a group that is n-ply transitive for all n has the same property. *Graham Higman.*

Szász, F. On cyclic groups. Fund. Math. **43** (1956), 238–240.

Making use of R. Baer's characterization of Hamiltonian groups [S.-B. Heidelberger Akad. Wiss. Math.-Nat. Kl. **1933**, no. 2, 12–17], the author proves that a group is cyclic if and only if its only subgroups are those of the form G^k, k a non-negative integer. *D. G. Higman.*

Moran, S. Basis for groups. Bull. Allahabad Univ. Math. Assoc. **16** (1951–56), 7–10.

Szász, F. Über Gruppen, deren sämtliche nicht-triviale Potenzen zyklische Untergruppen der Gruppe sind. Acta Sci. Math. Szeged **17** (1956), 83–84.

German version of Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **5** (1955), 491–492 [MR **17**, 709]. *A. Kertész.*

Cohn, P. M. A remark on the general product of two infinite cyclic groups. Arch. Math. **7** (1956), 94–99.

If a group G is factorizable into the product of two cyclic subgroups A and B, that is, $G=AB$ and $A\cap B=1$, then for every pair $a\in A$ and $b\in B$ there exist unique elements $a^{(b)}$ and $b^{(a)}$ in A and B, respectively, such that $ab=b^{(a)}a^{(b)}$. L. Rédei [Acta Math. Acad. Sci. Hungar. **1** (1950), 74–98; MR **14**, 13] had given a survey of all groups that can be factorized into the product of two cyclic subgroups provided at least one of them is infinite. His survey was not complete, because in the case when both subgroups are infinite, he had to make the additional assumption that there are elements $a\in A$ and $b\in B$, both distinct from the unit element, for which either $a^{(b)}=a$ or $b^{(a)}=b$. In the present paper the author proves that the case excluded by Rédei's assumption cannot occur, so that the survey is complete after all. The proof first reduces the problem to the case when the derived group G' of G has a trivial intersection with both A and B. Here some recent results of Itô [Math. Z. **62** (1955), 400–401; MR **17**, 125] are used. Next the author shows that the A-components of the elements of G' form a subgroup $\{x\}$ of A, and the B-components a subgroup $\{y\}$ of B and that the elements of G' are precisely x^ky^k. Now G' is known to be abelian (see Itô, loc. cit.), but the most difficult part of the proof is to show that G' is, in fact, cyclic. Once this is known, the proof is completed easily. *K. A. Hirsch.*

Meier-Wunderli, Heinrich. Über die Struktur der Burnsidegruppen mit zwei Erzeugenden und vom Primzahlexponenten $p>3$. Comment. Math. Helv. **30** (1956), 144–174.

The author continues the study [cf. same Comment. **26** (1952), 1–5; MR **13**, 818] of the Burnside group B, on two generators, with prime exponent p. The main tool is a refinement of Philip Hall's "collection process" [Proc. London Math. Soc. (2) **36** (1933), 29–95]. To state the principal result, let δ_w^p be the difference in dimension of the lower central quotient B_w/B_{w+1} from that of F_w/F_{w+1}, for F free on two generators; $\delta_{u,v}^p$ be the analogous defect for the homogeneous submodule of degrees u, v in the generators. Then: for $0\leqq\tau\leqq p-2$, $\delta_{3,p+\tau-3}^p=(\bar\tau+2\tau+3\tau^2)/3!$, where $\bar\tau=1, 2, 3, 4, -1, 6$ and $\bar\tau\equiv\tau$ (modulo 6). A corollary, that B has class $c\leqq 2p-1$, improves a result of J. A. Green [J. London Math. Soc. **27** (1952), 476–485; MR **14**, 350].

The Hall process yields a normal form for U in F as a product $\prod C^x$ of standard commutators C, descending in a prescribed order, together with information on the coefficients x, and, in particular, on the f as functions of the x, where $U^p\sim\prod C^f$. If C_1, C_2 are the free generators of F, then, modulo $F_2^pF_{2p-1}$, $U^p\equiv C_1^{px_1}C_2^{px_2}h(U)$, where the $h(U)$ lie in an abelian group of exponent p. In additive form, $h(U)=\sum m_i(x)e_i$, where the $m(x)$ are different monomials in $x_1, x_2, \cdots$, modulo p, and the e's are certain linear combinations of the C. The determination of the δ involves finding the number Π of a priori possible e's with leading term of prescribed degrees u, v in the generators, and then demonstrating the existence of U such that the leading terms of the $h(U)$ involve these e independently.

The foregoing is a very sketchy account of the first half of the paper, which contains many technical results of general interest. The second half is devoted to the detailed computations last mentioned above. (The reviewer noted several misprints of no serious consequence.)

R. C. Lyndon (Berkeley, Calif.).

Duguid, A. M.; and McLain, D. H. FC-nilpotent and FC-soluble groups. Proc. Cambridge Philos. Soc. **52** (1956), 391–398.

A group G which has a finite series of normal subgroups $G=G_1\geqq G_2\geqq\cdots\geqq G_n=1$ such that, for each $i=2, 3, \cdots, n$, no element of G_{i-1}/G_i has more than a finite number of conjugates in G/G_i is called FC-nilpotent. The main result is a generalisation to FC-nilpotent groups of some known properties of nilpotent groups. It is shown that, for FC-nilpotent groups, the properties Max, Max-n and FG are equivalent; and also that the properties Min and Min-n are equivalent. Here, Max and Max-n are the maximal conditions for all subgroups and for all normal subgroups, respectively; Min and Min-n are the analogous minimal conditions; and FG is the property of being finitely generated. It is also shown that the finitely generated FC-nilpotent groups are precisely the finite extensions of finitely generated nilpotent groups; and that the FC-soluble groups which satisfy Max are precisely the finite extensions of soluble groups satisfying Max. (A group G is called FC-soluble if it has a finite series of subgroups $G=G_1\geqq G_2\geqq\cdots\geqq G_n=1$, each normal in its predecessor, such that, for each $i=2, 3, \cdots, n$, no element of G_{i-1}/G_i has more than a finite number of conjugates in G_{i-1}/G_i.) Finally, it is shown that an FC-soluble group satisfying Min is a finite extension of a direct product of a finite number of groups of type C_{p^∞} (isomorphic with the group of all p^nth roots of unity, $n=1, 2, 3, \cdots$).

P. Hall (Cambridge, England).

Trofimov, P. I. Finite nilpotent groups with four classes of noninvariant conjugate subgroups. Tomskiĭ Gos. Univ. Uč. Zap. Mat. Meh. **25** (1955), 45–48. (Russian)

Let Γ be nilpotent and have exactly four classes C_1, C_2, C_3, C_4 of noninvariant conjugate subgroups. The number of distinct primes dividing the order of Γ is 2 or 3. If it is 3, then $\Gamma=\Pi\times\{S\}\times\{T\}$, where Π is defined in the following review, and $\{S\}$, $\{T\}$ have (different) prime orders. In case 2 primes divide $o(\Gamma)$, no more than two of the classes C_i can consist of groups of prime-power order. If only one class C_1 is of this type, then $o(\Gamma)=p^\alpha q^\beta$, $\Gamma=\Pi\times\{S\}$, $o(\{S\})=q^\beta$. If two classes C_1, C_2 are of this type, the subgroups in these classes must lie in the same Sylow subgroup M of Γ, and $\Gamma=M\times\{S\}$, $o(\{S\})=p\neq 2$, $o(M)=8$ or 16. In case $o(M)=8$, M is the group of symmetries of the square; if $o(M)=16$, M is defined by $A^4=B^4=C^4=ABAB^{-1}=ACA^{-1}C^{-1}=BCB^{-1}C^{-1}=I$, $A^2=B^2=C^2$. *J. L. Brenner* (Palo Alto, Calif.).

Trofimov, P. I. Finite nilpotent groups with a given number of classes of noninvariant subgroups. Tomskiĭ Gos. Univ. Uč. Zap. Mat. Meh. **25** (1955), 40–42. (Russian)

Let Π be the primary group of order p^α $(>p)$ defined by $P^e=S^p=1$, $PSP^{-1}=S^{p^f}$ $(e=p^{\alpha-1}, f=p^{\alpha-2})$, and let Λ be the primary cyclic group of order q^β $(p, q)=1)$. Then $\Pi\times\Lambda$ is nilpotent and has exactly $1+\beta$ classes of conjugate noninvariant subgroups. That is, (theorem 1), the number of such classes in a finite nilpotent group Γ can be any integer $\rho(\Gamma)$. Theorem 2. The inequality $\rho(\Gamma)\geqq 2^{\tau-1}$ holds if Γ is nilpotent but neither abelian or hamiltonian; here τ is the number of distinct prime divisors in the order of Γ. *J. L. Brenner* (Palo Alto, Calif.).

Zacher, Giovanni. Sull'ordine di un gruppo finito risolubile somma dei suoi sottogruppi di Sylow. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **20** (1956), 171–174.

The author proves that if a solvable group is the (set-theoretical) union of its Sylow subgroups, then its order is not divisible by more than two different prime numbers. He also finds two properties of subgroups of such a group. *H. A. Thurston* (Bristol).

Fletcher, T. J. Campanological groups. Amer. Math. Monthly **63** (1956), 619–626.

This paper gives a simple account of the group-theoretical basis of change-ringing as practised in England, with several examples. In connexion with W. H. Thompson's work on Grandsire Triples, the author states that it is difficult to see how Thompson's tedious enumeration of cosets can be avoided. {This enumeration can, in fact, be avoided, not only for this method of ringing changes, but also in more general situations, as was shown by the reviewer [Proc. Cambridge Philos. Soc. **44** (1948), 17–25; MR **9**, 267]. Since the author denotes changes by multiplication in the left his operator $B^{-1}C$ should be replaced by CB^{-1} throughout; this does not alter the permutations in the given numerical cases.} *R. A. Rankin.*

Higman, Graham. On finite groups of exponent five. Proc. Cambridge Philos. Soc. **52** (1956), 381–390.

It is shown that the order of a finite group of exponent 5 which can be generated by k elements, k finite, is bounded by a number depending only on k. This establishes the truth of the restricted Burnside conjecture for groups of exponent 5. The paper is independent of the work of A. I. Kostrikin [Izv. Akad. Nauk SSSR. Ser. Mat. **19** (1955), 233–244; MR **17**, 126], which deals only with the case $k=2$. The proof is obtained from a theorem on associative rings. Let an associative ring R be generated by elements x, a, b, $\cdots$; and let R_0 be the Lie ring generated by these same elements, using the usual bracket multiplication $[u, v]=uv-vu$. If $6y=0$, $y\in R$ implies $y=0$ and if $u^4=0$ for all $u\in R_0$, then there exists an integer N such that every monomial in x, a, b, $\cdots$ of degree than N in x is zero. With the help of this theorem it is shown that a finite group G of exponent 5 with k generators has class at most Nk, from which the boundedness of the order of G follows at once. *P. Hall.*

Carlitz, L. Resolvents of certain linear groups in a finite field. Canad. J. Math. **8** (1956), 568–579.

Let Γ denote the group of linear transformations $x'=(ax+b)/(cx+d)$, $(ad-bc=1)$ with coefficients in the field $F_q=\mathrm{GF}(q)$. It is known that Γ possesses an absolute fundamental invariant $J=J(x)$ which, for odd q, can be expressed as $J=Q^{\frac{1}{2}(q+1)}L^{-(q^2-\frac{1}{2}q)}$, where $L=x^q-x$ and $Q=L^{q-1}+1$ whilst, for even q, $J=Q^{q+1}L^{-(q^2-q)}$. The equation $J(x)=y$, where y is an indeterminate, is normal over $F_q(y)$ with Galois group Γ. A resolvent of this equation of degree $q+1$ can immediately be written down as follows: $u=L^{\frac{1}{2}(q-1)}$, $(u^2+1)^{\frac{1}{2}(q+1)}=yu^q$ (q odd); $u=L^{q-1}$, $(u+1)^{q+1}=yu^q$ (q even). The main object of the paper is to obtain resolvents of lower degree if they exist. Now it is known that Γ can be represented as a permutation group of degree $\leqq q$ only when $q=5, 7, 9, 11$. In these cases the author explicitly constructs resolvents of degrees 5, 7, 6, 11 respectively. The paper also contains a discussion of a particular equation of degree 8 whose Galois group is the ternary group LF(3, 2) of order 168. *W. Ledermann* (Manchester).

Edge, W. L. The characters of the cubic surface group. Proc. Roy. Soc. London Ser. A. **237** (1956), 132–147.

Continuing his previous study of the cubic surface group G of order 51840 [same Proc. **228** (1955), 129–146; **233** (1955), 126–146; MR **16**, 1046; **17**, 941], the author uses a geometrical method, based on the 5-rowed orthogonal representation of G over GF(3), to obtain the characters of 27 reducible induced representations of G, none with more than 3 irreducible components. These are induced by certain geometrically significant subgroups of G of index 27, 36, 40 and 45. From linear combinations of these all but one of the 25 irreducible characters of G are found, and the missing one is obtained from orthogonality. For example, not only are 27 pentagons in PG(4, 3) permuted by G, but their vertices suitably ordered are permuted by the operations of the symmetric group S_5, six of whose irreducible representations induce "useful" representations of G. The characters obtained for G agree with those previously obtained by Frame [Ann. Mat. Pura Appl. (4) **32** (1951), 83–119; MR **13**, 817]. *J. S. Frame* (E. Lansing, Mich.).

Taketa, Kiyosi. Über die Struktur der metabelschen Gruppen. IV. J. Math. Soc. Japan **7** (1955), 491–529.

Continuing a series of three previous articles [Jap. J. Math. **13** (1937), 129–232; J. Osaka Inst. Sci. Tech. Part I. **2** (1950), 1–28; Tôhoku Math. J. (2) **4** (1952), 10–32; MR **15**, 285] the author studies metabelian groups by a detailed analysis of canonical forms for representing their maximal abelian p-subgroups by triangular matrices over a Galois field $\mathrm{GF}(p^s)$. Clearly the maximal abelian sub-

group A of G is a direct product of abelian p-groups. Furthermore, any element of GF(p^{sm}) can be written as an $m\times m$ matrix over GF(p^s). Thus the triangular p-matrix with unit matrix subtracted is broken up into subdiagonal blocks whose linear dependence is analyzed to achieve a canonical form with many vanishing parts. The canonical form gives insight into the group structure. *J. S. Frame* (E. Lansing, Mich.).

Coxeter, H. S. M. The collineation groups of the finite affine and projective planes with four lines through each point. Abh. Math. Sem. Univ. Hamburg **20** (1956), 165–177.

The group of all collineations of the finite projective plane PG(2, 3) is the simple group LF(3, 3) of order 5616. In this paper a complete set of defining relations for this group is found, these are $S^6=T^3=(ST)^4=(S^2T)^4=(S^3T)^3=E$, $(TS^2T)^2S^3=S^3(TS^2T)^2$. This has a subgroup of index 26 generated by T and $U=S^2$, the subgroup being the Hessian group of order 216, this being the group of projective collineations leaving invariant the nine inflection points of the general cubic curve in the complex projective plane. The Hessian group has a subgroup of index 9 which is the binary tetrahedral group of order 24. The treatment throughout is in geometric terms.
Marshall Hall, Jr. (Columbus, Ohio).

Pyateckiĭ-Šapiro, I. I. Classification of modular groups. Dokl. Akad. Nauk SSSR (N.S.) **110** (1956), 19–22. (Russian)

The author specializes the classification begun in his earlier work [same Dokl. (N.S.) **106** (1956), 973–976; MR **18**, 19] to algebraic Riemann matrices, yielding three types of multiplier algebras: The first type is a commutative algebra based on an imaginary quadratic extension of a totally real field. The second type has basis

$$1,\ i,\ j,\ k=ij,\ i^2=\alpha,\ j^2=\beta,\ ij=-ji,$$

and the third type has basis

$$1,\ i,\ j,\ k=ij,\ i^2=-\alpha,\ j^2=-\beta,\ ij=-ji,$$

where α, β are totally positive elements of the field.
H. Cohn (St. Louis. Mo.).

Kacman, A. D. On some properties of a semigroup invariant in a group. Uspehi Mat. Nauk (N.S.) **11** (1956), 179–183. (Russian)

Let S be an invariant semigroup in a torsion free group G. A subsemigroup A of S is said to be of periodic index in S if $g\neq 1$ and $g\in S$ implies $g^n\in A$ for some n. It is shown that the set of all elements of S, generating principal left ideals of periodic index, form an invariant subsemigroup of S. The set of all elements of S, not generating left ideals of periodic index in S, form an invariant prime ideal in S.

A necessary and sufficient condition that all left ideals of an integral semigroup S (invariant in a group G) possess periodic index is determined and the following analogue of lattice ordered groups is obtained: If all left ideals Sa, $a\in S$, possess periodic index in S and if the commutator of any pair of elements in S is contained in $S\cup S^{-1}$, then the semigroup S is commutative.
L. J. Paige (Los Angeles, Calif.).

Ponizovskiĭ, I. S. On matrix representations of associative systems. Mat. Sb. N.S. **38** (**80**) (1956), 241–260. (Russian)

Let S be a finite semigroup (associative system). Let P be a field, and let $\mathfrak{A}(S, P)$ be the semigroup algebra consisting of all formal linear combinations

$$\sum \alpha_x x\ (\alpha_x\in P,\ x\in S),$$

with termwise addition and scalar multiplication, and with $(\sum \alpha_x x)(\sum \beta_y y)=\sum\sum \alpha_x\beta_y xy$. If S has a 2-sided zero z, let $\mathfrak{A}_z(S, P)$ be the difference algebra

$$\mathfrak{A}(S, P)-\{\alpha z:\alpha\in P\}.$$

If S has no 2-sided zero, let $\mathfrak{A}_z(S, P)=\mathfrak{A}(S, P)$. Let K be the complex number field. The algebra $\mathfrak{A}(S, K)$ has been studied by Hewitt and Zuckerman [Acta Math. **93** (1955), 67–119; MR **17**, 1048], and an infinite dimensional analogue of $\mathfrak{A}(S, K)$ has also been analyzed by Hewitt and Zucker man [Trans. Amer. Math. Soc. **83** (1956), 70–97]. The algebra $\mathfrak{A}_z(S, P)$ has been closely studied by Munn [Proc. Cambridge Philos. Soc. **51** (1955), 1–15; MR **16**, 561]. It is easy to see that $\mathfrak{A}_z(S, P)$ is semisimple if and only if every representation of S by matrices over P that carries z into 0 is completely reducible. The paper under review is stated in terms of this latter property. The author first proves Munn's theorems (loc. cit. Lemma 3.3 and Theorem 4.7) on the semisimplicity of $\mathfrak{A}_z(S, P)$. (He states that the results were obtained in 1952 and 1953, before the publication of Munn's work.)

The author next takes up various special semigroups. Let Σ_n be the semigroup of all "partial transformations" of a set T consisting of n elements into or onto itself: that is, the set of all one-to-one transformations with domain and range both contained in T, and with ordinary iteration as the semigroup operation. The transformation with void domain is the zero of Σ_n. Theorem 3. $\mathfrak{A}_z(\Sigma_n, P)$ is semisimple if P has characteristic zero. If P has characteristic $p>0$, then $\mathfrak{A}_z(\Sigma_n, P)$ is semisimple if and only if $n<p$. (This result is easy to prove from Theorem 9.5 of Munn, loc. cit.) Theorem 4. Let H be a subsemigroup of Σ_n that is closed under the formation of inverses. Then $\mathfrak{A}_z(H, P)$ is semisimple if P has characteristic zero. If P has characteristic $p>0$, then $\mathfrak{A}_z(H, P)$ is semisimple if and only if $x\in H$ and $x^{p+1}=x$ imply $x^2=x$. Theorem 6. Suppose that the least 2-sided ideal of S is neither $\{z\}$ nor a group. Then $\mathfrak{A}_z(S, P)$ is semisimple for no choice of P. Theorem 8. Let S be the union of groups, and let $S=J_0\supset J_1\supset\cdots\supset J_w\supset\emptyset$ $(J_0\neq J_1\neq\cdots\neq J_w\neq\emptyset)$ be a composition series for S in the sense of Rees [ibid. **36** (1940), 387–400; MR **2**, 127]. Then $\mathfrak{A}_z(S, P)$ is semisimple if and only if (1) J_i-J_{i+1} is a group $(i=0, \cdots, w-1)$ and (2) $x^{p+1}=x$ in S implies $x^2=x$, if P has characteristic p. In a final paragraph, the author gives a general method for finding all irreducible representations of S by matrices over P, in the case where P is algebraically closed and $\mathfrak{A}_z(S, P)$ is semisimple. *E. Hewitt* (Seattle, Wash.).

Bredihin, B. M. An example of a finite homomorphism with a bounded summation function. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 4(70), 119–122. (Russian)

Let G be a semigroup of real numbers $\geqq 1$ with a finite or countably infinite basis. Let Z be a finite set of n non-zero complex numbers. A finite homomorphism is a homomorphic mapping $\alpha\in G\rightarrow\chi(\alpha)\in Z$. (Clearly $\chi(\alpha)=\exp(2\pi ik/n)$ with a $0\leqq k<n$). $\chi(\alpha)$ is called a generalized character if the summation function $\sum_{\alpha\leqq x,\alpha\in G}\chi(\alpha)$ is bounded. (The Dirichlet characters are generalized characters of semigroups of positive integers with a basis containing almost all primes.)

It is proved: Let G be a semigroup of rational numbers $r\geqq 1$ with a finite basis $r_1, \cdots, r_N$ $(N\geqq 2)$ and $\chi(r)$ a finite

homomorphism of G. Then there exists a semigroup of rational numbers $G_0 \supset G$ having an infinite basis and an extension $\chi_0(r)$ or $\chi(r)$ to the semigroup G_0 such that $\chi_0(r)$ is a generalized character.

The author remarks that this result can be regarded as a first step in the effort to solve the following problem due to Čudakov: does there exist a generalized character of a semigroup of positive integers which is different from a Dirichlet character?

Š. Schwarz (Bratislava).

Popova, Helen. Logarithmetics of finite quasigroups. II. Proc. Edinburgh Math. Soc. (2) **9** (1956), 109–115.

The author continues her study of the logarithmetic L_Q of a finite quasigroup Q. [For definitions and preliminary results see part I, same Proc. (2) **9** (1954), 74–81; MR **16**, 564.] Several results concerning the structure and order of the finite quasigroup $L_Q(+)$ are obtained. The most interesting of these is that if Q is plain (i.e. has no subquasigroups other than itself) the order of $L_Q(+)$ is a power of the order of Q.

D. C. Murdoch.

See also: Mal'cev, p. 370; Jenner, p. 375; Balcerzyk and Mycielski, p. 403; Walker, p. 403.

THEORY OF NUMBERS

General Theory of Numbers

Salmeri, Antonio. Risoluzione in numeri interi di una particolare equazione fratta con numero indeterminato di variabili. Period. Mat. (4) **34** (1956), 234–239.

For fixed n, x_0, a (throughout most of the paper a is taken equal to 10) find two numbers with n digits in the scale of a whose quotient is x_0, under the condition that if the numerator has the form $x_n, x_{n-1}, \cdots, x_2, x_1$, then the denominator has the form $x_{n-1}, x_{n-2}, \cdots, x_1, x_n$.

Elston, Fred G. A generalization of Wilson's theorem. Math. Mag. **30** (1957), 159–162.

The generalization (for Wilson's theorem $r=0$) is that, for $r=0, 1, 2, \cdots, m$, each one of the $(m+1)$ conditions $r!(n-r)!\equiv(-1)^{r+1} \pmod p$ is necessary and sufficient for the primality of p, where $n=p-1$ and $m=[\frac{1}{2}n]$.

Levit, R. J. A minimum solution of a diophantine equation. Amer. Math. Monthly **63** (1956), 646–651.

It is traditional in treating the solution of the linear diophantine equation $ax+by=c$ to consider first the equation $a\xi+b\eta=1$. Once this is solved by a continued fraction algorithm, ξ and η are multiplied by c to give x, y. This often results in unnecessarily large values. The author proposes a variant of the traditional process in which a sequence of c's is obtained and which leads directly to minimum solutions of $ax+by=c$. In its "absolutely least" form the algorithm produces two sets of partial quotients (b_i, c_i) defined by $b_0=a$, $b_1=b$, $c_0=c$, $b_{i-1}=q_ib_i+b_{i+1}$, $c_{i-1}=g_ib_i+c_i$, where $|b_{i+1}|$ and $|c_i|$ do not exceed $\frac{1}{2}|b_i|$. The process terminates at $i=n$, where for the first time either $c_n=0$ or $c_n=\pm\frac{1}{2}b_n$. Following Gauss, a single sequence x_k now gives both $x=x_1$ and $y=x_0$. The recurrence is

$$x_{i-1}=g_i-q_ix_i+x_{i+1},$$

where intially $x_n=c_n$ and $x_{n+1}=-\operatorname{sgn} c_n[\frac{1}{2}b_{n+1}]$. Conditions on c for the existence of solutions (x, y) in which $|x|$ and $|y|$ both attain their least possible values are found. Such solutions when they exist are onbtaied by the algorithm.

D. H. Lehmer (Berkeley, Calif.).

Carlitz, L. Note on a quartic congruence. Amer. Math. Monthly **63** (1956), 569–571.

The object of this note is to obtain the factorization of the quartic polynomial $x^4+mx^2+n \pmod p$ into its linear and irreducible quadratic factors. Here, p is an odd prime and m and n are rational integers. The method used is strictly elementary without reference to groups or algebraic number fields.

I. A. Barnett.

Parameswaran, S. Number of recurring cycles. Monatsh. Math. **60** (1956), 183–189.

Let $(N, 10)=1$. The fractions M/N $(1\leq M<N)$ generate $N-1$ pure recurring decimal fractions. Two cycles are said to be distinct if one is not the cyclic permutation of the other. If N is a power of a prime it is known that the number of distinct cycles generated by M/N $(1\leq M<N)$ equals $(N-1)/d$ where d is the least number >0 satisfying $10^d\equiv1 \pmod N$. The author obtains a formula for the number of distinct cycles for every N, $(N, 10)=1$, but his formula is fairly complicated and is in terms of the prime factor decomposition of N.

P. Erdős.

Harris, V. C. Another proof of the infinitude of primes. Amer. Math. Monthly **63** (1956), 711.

Rossi, Francesco Saverio. I numeri inversi all'unità rispetto al modulo M (intero positivo). Archimede **8** (1956), 279–281.

Salié, Hans. Über die Koeffizienten der Blasiusschen Reihen. Math. Nachr. **14** (1955), 241–248 (1956).

Let $y(x)$ be the solution of the differential equation $y^{(n)}=\lambda yy^{(n-1)}$ with $y=y'=\cdots=y^{(n-2)}=0$, $y^{(n-1)}=1$ at $x=0$. $y(x)$ has a series expansion

$$\sum_1^\infty \lambda^{k-1}x^{nk-1}((nk-1)!)^{-1}c_k^{(n)}.$$

The numbers $c_k^{(n)}$ are always integers. The author obtains several numbertheoretic properties of the numbers $c_k^{(n)}$. The results include: (1) For fixed k,

$$\lim_{n\to\infty} n^{(k-1)/2}k^{-kn}c_{k+1}^{(n)}=k^{1/2}(2\pi)^{(1-k)/2}(k!)^{-1}.$$

(2) If p is a prime and $p|n$ with $n=pn_1$, then $c_k^{(n)}\equiv c_k^{(n_1)} \bmod p$. (3) If p is a prime, $p>2$, $c_k^{(p+1)}\equiv k!(k-1)!2^{1-k} \bmod p$, so that $c_k^{(p+1)}\equiv0 \bmod p$ for $k\geq p>2$. The author indicates without proofs sharper results in certain cases.

W. S. Loud (Minneapolis, Minn.).

Hemer, Ove. On some diophantine equations of the type $y^2-f^2=x^3$. Math. Scand. **4** (1956), 95–107.

This paper is based on some previous work by the same author [Ark. Mat. **3** (1954), 67–77; MR **15**, 776; see also L. J. Mordell, Proc. London Math. Soc. (2) **13** (1913), 60–80; T. Nagell, Math. Z. **28** (1928), 10–29]. If the cubic form $AU^3+BU^2V+CUV^2+DV^3$ is denoted by (A, B, C, D) then, as the author has previously shown, an equation $y^2-f^2=x^3$ may be replaced by the equation $(1, 0, 0, 1)=2f$ and $\frac{1}{2}(3^r-1)$ equations $(A, 0, 0, D)=2fA^{-1}D^{-1}$, if $2f$ contains r distinct primes. Using the methods of algebraic

numbers the author finds all the solutions of the equations $y^2-f^2=x^3$ for the fifteen values of $f=11, 12, \cdots, 25$.

I. A. Barnett (Cincinnati, Ohio).

Koksma, J. F. Sur les suites $(\lambda_n x)$ et les fonctions $g(t)\in L^{(2)}$. J. Math. Pures Appl. (9) **35** (1956), 289–296.

Let $0<\lambda_1<\lambda_2<\cdots$ be a sequence of integers. The author investigates conditions which would insure that for every $g(t)\in L^{(2)}$ of period 1

$$(1)\qquad \frac{1}{N}\sum_{k=1}^{N} g(\lambda_k x)\to\int_0^1 g(t)dt,$$

holds for almost all x.

The author proved [Nederl. Akad. Wetensch., Proc. **53** (1950), 959–972; MR **12**, 86] that if $\sum_{h=2}^{\infty}|c_h{}^2|\log h<\infty$ then (1) holds, for almost all x where the c_h are the Fourier coefficients of $g(t)$. The reviewer proved [Trans. Amer. Math. Soc. **67** (1949), 51–56; MR **11**, 375] that if $\lambda_{k+1}>(1+c)\lambda_k$ then there exists a $g(t)$ for which (1) does not hold, in fact for almost all x

$$\limsup \frac{1}{N}\sum_{k=1}^{N} g(\lambda_k x)=\infty.$$

But if $\lambda_{k+1}>(1+c)\lambda_k$ and $\sum_{h=2}^{\infty}|c_h{}^2|$ $(\log\log h)^2<\infty$, then (1) holds for almost all x. The author proved [Bull. Soc. Math. Belg. **6** (1953), 4–13; MR **16**, 682], that if $\lambda_n=n$ and $\sum_{h=2}^{\infty}|c_h{}^2|\log\log h<\infty$ then (1) holds for almost all x.

In the present note the author sharpens his result of 1950 cited above for some sequences λ_k. His results are fairly complicated, we only state here two special cases: 1) If $\lambda_n=n^k$ and $\sum_{h=2}^{\infty}|c_h|^2\sum_{d^k|h}\frac{1}{d}<\infty$, then (1) holds for almost all x; 2) if $(\lambda_m,\lambda_n)=1$ and

$$\sum_{h=2}^{\infty}|c_h{}^2|\sum_{\lambda_n|h}\frac{1}{n}<\infty,$$

then (1) holds for almost all x. Clearly there are still many interesting and unsolved problems here.

P. Erdös (Birmingham).

Steiger, Franz. Über die Grundlösungen der Gleichung $a^2+b^2+c^2=d^2$. Elem. Math. **11** (1956), 105–108.

A set of formulas involving four integral parameters is announced for the solutions in integers of $a^2+b^2+c^2=d^2$. The formulas give each fundamental solution (such that $(a,b,c,d)=1$) exactly once. Proofs will be given later.

I. Niven (Eugene, Ore.).

Cassels, J. W. S. Addendum to the paper "Bounds for the least solutions of homogeneous quadratic equations". Proc. Cambridge Philos. Soc. **52** (1956), 602.

In the bound obtained by the writer [same Proc. **51** (1955), 262–264; MR **16**, 1002] for the least solution in integers of homogeneous quadratic equations in n variables, there appeared significantly the exponent $\frac{1}{2}(n-1)$, which is now shown to be best possible. *I. Niven.*

Gloden, A. Construction de systèmes multigrades remarquables. Mathesis **65** (1956), 230–234.

Identities are given for obtaining equal sums of like powers of integers. *I. Niven* (Eugene, Ore.).

Manin, Yu. I. On cubic congruences to a prime modulus. Izv. Akad. Nauk SSSR. Ser. Mat. **20** (1956), 673–678. (Russian)

Let $p>0$ be prime, k_p the field of p elements and a, b elements of k_p with $4a^3\neq 27b^2$. The author shows that the number N of solutions $x, y\in k_p$ of

$$(1)\qquad y^2=x^3+ax+b$$

satisfies

$$|N-p|<2p^{1/2}.$$

This is, of course, a particular case of the Riemann hypothesis for function fields [H. Hasse, J. Reine Angew. Math. **175** (1936), 55–62, 69–88, 193–208; A. Weil, Sur les courbes algébriques et les variétés qui s'en déduisent, Hermann, Paris, 1948; MR **10**, 262] but the author's proof is entirely elementary in its execution, though admittedly motivated by Hasse's ideas. The author points out that his proof can be extended to any function field over a finite field.

Departing somewhat from the author's notation for convenience consider the curve

$$(2)\qquad X^3+aX+b=(x^3+ax+b)Y^2$$

over the field $k_p(x)$ of rational functions of an indeterminate x. Two points on (2) with coordinates in $k_p(x)$ are $\mathfrak{p}$: $X=x$, $Y=1$ and $\mathfrak{P}$: $X=x^p$, $Y=(x^3+ax+b)^{\frac{1}{2}(p-1)}$. Define addition of points on (2) in the usual way with respect to the improper point $\mathfrak{O}$ corresponding to the point at infinity in the characteristic 0 case. For integer n put $d_n=0$ if $\mathfrak{P}+n\mathfrak{p}=\mathfrak{O}$. Otherwise d_n is to be the degree of the numerator of the X-coordinate of $\mathfrak{P}+n\mathfrak{p}$ when it is expressed as a quotient of polynomials in x with no common factor. The proof now runs in three steps. (I) $d_{-1}-d_0-1=N-p$. For

$$X_{-1}=-x-x^p+\frac{\{1+(x^3+ax+b)^{\frac{1}{2}(p-1)}\}^2(x^3+ax+b)}{(x-x^p)^2}.$$

Now $x^p-x=\prod_{j\in k_p}(x-j)$ and $(j^3+aj+b)^{\frac{1}{2}(p-1)}$ is the quadratic residue symbol of j^3+aj+b. Hence a factor $x-j$ divides $\{1+(x^3+ax+b)^{\frac{1}{2}(p-1)}\}$ if and only if j^3+aj+b is not a square in k_p. The result (I) now follows easily. (II) $d_{n-1}+d_{n+1}=2d_n+2$. If one of $\mathfrak{P}+n\mathfrak{p}$, $\mathfrak{P}+(n\pm 1)\mathfrak{p}$ is $\mathfrak{O}$, then (II) is trivial. Otherwise, let the x-coordinate of $\mathfrak{P}+n\mathfrak{p}$ be A_n/C_n, where A_n, C_n are polynomials with no common factor. By the formulae for addition we have

$$(3)\qquad \frac{A_{n-1}}{C_{n-1}}\,\frac{A_{n+1}}{C_{n+1}}=\frac{(A_nx-aC_n)^2-4bC_n(C_nx+A_n)}{(C_nx-A_n)^2}.$$

A detailed consideration of common factors shows that the numerator and denominator of (3) are equal; (II) now follows from the equality of the numerators since $d_n=$ degree A_n and it is not difficult to see that degree $C_n<$ degree A_n. (III) From (I), (II) we have

$$d_n=n^2-(N-p)n+p.$$

But $d_n\geq 0$ by definition and if $d_n=0$ then $d_{n\pm 1}>0$. Hence $(N-p)^2-4p\leq 0$: so $|N-p|<2p^{\frac{1}{2}}$, as asserted.

Postscript: Mr. B. J. Birch has pointed out to me that it is by no means obvious that the degree of A_n is greater than that of C_n, and that the author's proof of this contains an error on line 23 of page 675. Indeed it is easy to construct curves (2) with points (X, Y) where the degree of the numerator of X is less than that of the denominator, though I cannot show that they are thrown up by the author's construction of X_n. However the author's proof can be salvaged with some modifications by de-

fining d_n to be the maximum of the degree of A_n and C_n. Closer examination shows that the author's proof can be considerably shortened, much of the detailed case-by-case discussion being unnecessary. *J. W. S. Cassels.*

Gabard, E. Quelques factorisations. Mathesis **65** (1956), 415–416.

This note gives a half-dozen factorizations of numbers of the form $k\cdot 2^n+1$ for $n=19$, 23 and 26. There are two misstatements about the factors of 2^n+1 for very large n. The author verifies the primality of $2^{21}3^4+1=12192^2+4607^2$. *D. H. Lehmer* (Berkeley, Calif.).

Remorov, P. N. On Kummer's theorem. Leningrad. Gos. Univ. Uč. Zap. **144.** Ser. Mat. Nauk **23** (1952), 26–34. (Russian)

The author forms estimates for the magnitude of primes p associated with a counter-example to Fermat's last theorem, i.e., $x^p+y^p=z^p$, $(p, xyz)=1$. The estimates are of the type $N_k<p<M_k$ where k is the degree of irregularity, i.e., $p^k|h_1$, $p^{k+1}\nmid h_1$, for h_1 the first factor of the class number of $R(\exp 2\pi i/p)$. [See Vandiver, Bull. Nat. Res. Council no. **62** (1928), 28–111]. The value of M_k, (or the fact that $\lim k=\infty$, as $p\to\infty$), follows from results of Vandiver and Kummer [op. cit., p. 85] that

$$h_1\equiv\pm p\prod B_{\frac{1}{2}(sp^K+1)}2^{-\frac{1}{2}(p-3)} \pmod{p^K},$$

whereas, as $p\to\infty$, more and more of these B_n are necessarily divisible by p, [op. cit., p. 65]. The value of N_k ($=2k+$const) follows even more simply from formulas: $p^k\leq h_1\leq(2p)^{-\frac{1}{2}(p-3)}\prod\sum_j$. [op. cit., p. 35]. No discussion of numerical data is given. *H. Cohn* (St. Louis, Mo.).

Remorov, P. N. On indeterminate equations of form $a^p+Db^p=c^p$. Dokl. Akad. Nauk SSSR (N.S.) **106** (1956), 395–398. (Russian)

The author considers the values of the prime $p\geq 3$ and the integer D such that a non-trivial integer solution of $x^p+y^p=Dz^p$ exists and generalizes results about the Fermat case $D=1$. He finds it necessary to restrict D by the condition $\sum f_j^{-1}<1$, where the f_j are the exponents to which the distinct prime divisors of D belong to modulus p. The following results are stated and proofs of (I), (II), (V) are sketched.

$$\text{(I)}\quad \begin{aligned}&\left[\frac{d^{p-1}}{dv^{p-1}}\log(x+e^v y)\right]_{v=0}\equiv\frac{(x+y)^{p-1}-1}{p}\ (p),\\ &B_n\left[\frac{d^{p-2n}}{dv^{p-2n}}\log(x+e^v y)\right]_{v=0}\equiv 0\ (p)\ (1\leq n\leq\tfrac{1}{2}(p-1))\end{aligned}$$

[cf. Kummer, Abh. Akad. Wiss. Berlin **1857**, Math. Abh., 41–74]. (II) If $x\not\equiv y$ (p), then

$$3^p-3\equiv\frac{5p}{2}\sum_{k=0}^{p-1}(-1)^k k^{p-2}\ (p^2).$$

(III) If $x\equiv y$ (p), then

$$2^{p-1}\equiv D^{p-1}\ (p^2)$$

[cf. Frobenius, S.-B. Preuss. Akad. Wiss. **1914**, 653–681]. (IV) There is an N_k such that if $p>N_k$ and the first factor of the class-number of the pth cyclotomic field is exactly divisible by p^k, then $Dxyz(x-y)\equiv 0$ (p) [see the paper reviewed above]. (V) If $x\equiv 0$ (r), $r\not\equiv 1$ (p), then $r^{p-1}\equiv 1$ (p^2). *J. W. S. Cassels.*

Noguera, Rodrigo. Unpublished mathematical investigations. Stvdia. Rev. Univ. Atlantico **1** (1956), nos. 3-4-5, 107–148. (Spanish)

A series of discussions, with historical emphasis, of the solution of the cubic equation. One article deals with the identity of Sophie Germain (also named after Euler) arising in the study of Fermat's last theorem.

Succi, Francesco. Sulla espressione del quoziente integrale di due funzioni aritmetiche. Rend. Mat. e Appl. (5) **15** (1956), 80–92.

The divisor calculus product $f\cdot g$ of two numerical functions f and g being defined as the function h for which

$$(1)\qquad h(n)=\sum_{\delta|n}f(\delta)g(n/\delta),$$

the summation extending over all the divisors δ of n, the author considers the problem of finding f when g and h are given. When $g(1)\neq 0$ the problem is solved uniquely by $f=g^{-1}\cdot h$. If $g(1)=0$ the inverse, g^{-1}, does not exist and f exists for only certain h, and g. Formulas are obtained for f by writing down the appropriate instances of (1) and "solving" the resulting linear equations by Cramer's rule. *D. H. Lehmer* (Berkeley, Calif.).

See also: Jenner, p. 375; Cohen, p. 382; Pagni, p. 382; Lombardo-Radice, p. 411.

Analytic Theory of Numbers

Vinogradov, I. M. Special cases of estimations of trigonometric sums. Izv. Akad. Nauk SSSR. Ser. Mat. **20** (1956), 289–302. (Russian)

This paper assumes a knowledge of both the results and the methods of proof of two earlier papers [same Izv. **12** (1948), 225–248; **15** (1951), 109–130; MR **10**, 599; **13**, 338]. Because of the complexity of the subject, the reviewer is unable to do more than state the results. Let $f(x)=A_nx^n+\cdots+A_1x$ be a real polynomial of fixed degree $n\geq 12$, and let m be an integer. Let

$$S_m=\sum_{x=1}^{P}\exp\{2\pi i\,m\,f(x)\},\quad T_m=\sum_{p\leq P}\exp\{2\pi i\,m\,f(p)\},$$

where p runs through the primes. Let $\nu=1/n$.

Theorem 1. Let $\tau_1=P^{1/3}$, $\tau_s=P^{s-(1-\nu)/3}$ for $s=2,\cdots,n$. Approximate to $A_n,\cdots,A_1$ by $|A_s-a_s/q_s|<(q_s\tau_s)^{-1}$, $0<q_s\leq\tau_s$, $(a_s, q_s)=1$. Let $Q=\mathrm{LCM}(q_n,\cdots,q_2)$ and $Q_1=\mathrm{LCM}(q_n,\cdots,q_1)$. Then (a) if $Q\geq P^{(1-\nu)/3}$ and $0<m\leq P^{\nu(1-\nu)/60}$ we have $S_m=O(P^{1-\rho})$, where $\rho^{-1}=10n^2\log 40n^2$; (b) if $Q<P^{(1-\nu)/3}$ but $Q_1>Q$ we have $S_1=O(P^{(2-\nu)/3})$; (c) if $Q<P^{(1-\nu)/3}$ but $Q_1=Q$, and if

$$\mu_0=\max(1, P^n|A_n-a_n/q_n|,\cdots,P|A_1-a_1/q_1|),$$

we have $S_1=O(PQ^{-\nu+\varepsilon}\mu_0^{-\nu})$ for any fixed $\varepsilon>0$.

Theorem 2. Approximate to $A_n,\cdots,A_2$ as above but with $\tau_s=P^{s/2}$ for $s=2,\cdots,n$. Suppose that some of $q_n,\cdots,q_2$ do not exceed $P^{1/4}$ and let Q denote the LCM of any selection from them. Define κ by $\kappa=\frac{1}{4}$ if $Q>P^{1/4}$ and by $Q=P^\kappa$ otherwise; define ρ_0 by

$$\rho_0=\kappa/\{6.7n^2\log 12n^2\}.$$

Then if $0<m\leq P^{2\rho_0}$ we have $T_m=O(P^{1-\rho_0+\varepsilon})$.

Theorem 3. Approximate to $A_n,\cdots,A_1$ as above with $\tau_s=P^{s/2}$ for $s=1,\cdots,n$. Define Q and Q_1 in the same way as in Theorem 1 and suppose that $Q\leq P^{1/9}$. Define κ_1 by $Q_1=P^{\kappa_1}$ and define ρ_1 by

$$\rho_1=\min\left(\frac{\kappa_1}{6.75n^2\log 12n^2},\ \frac{1}{27n^2\log 108n^2}\right).$$

Then if $0<m\leq P^{2\rho_1}$ we have $T_m=O(P^{-1\rho_1+\varepsilon})$. *H. Davenport* (London).

Kopřiva, Jiří. Remark on the significance of the Farey series in number theory. Publ. Fac. Sci. Univ. Masaryk **1955**, 267–279. (Czech. Russian summary)

M. Mikolás [C. R. Acad. Sci. Paris **228** (1949), 633–636; Acta Sci. Math. Szeged **13** (1949), 93–117; **14** (1951), 5–21; MR **10**, 433; **11**, 645; **13**, 627, 1138] has shown that Riemann's hypothesis about the zeros of $\zeta(s)$ is equivalent to the truth of the relation

$$\text{(A)} \qquad \sum_{\nu=1}^{P(x)} f(\rho_\nu)-P(x)\int_0^1 f(x)dx=O(x^{\frac{1}{2}+\varepsilon})$$

for every $\varepsilon>0$; here ρ_ν runs through the Farey fractions of order $[x]$ which lie in the interval $(0, 1)$, $P(x)$ is the number of such Farey fractions, and $f(\xi)$ may be $\sin \lambda\xi$, $\cos \lambda\xi$ $(0<|\lambda|<2\{(\zeta(3)+5\pi^{-2})/5\}^{-\frac{1}{2}}, |\lambda|\neq\pi)$, a quadratic polynomial, or a cubic polynomial $a_0\xi^3+a_1\xi^2+a_2\xi+a_3$ $(a_1\neq 3a_0/2)$. The function f may take other forms, but those are not considered in the present paper. Supposing f to be any one of the four types mentioned above, the author shows that if from the sum on the left of (A) are omitted those terms which correspond to Farey fractions of type $a/2^n$ and $P(x)$ is reduced accordingly, relation (A), thus altered, is still equivalent to Riemann's hypothesis. He proves a similar result in which the excluded Farey fractions are of type b/m^2, and he asks how far one may carry on this process of exclusion without disturbing the equivalence between the corresponding relations derived from (A), and Riemann's hypothesis.

H. Halberstam (Exeter).

Golubev, V. A. Généralisations du théorème de Dirichlet sur les nombres premiers. Mathesis **65** (1956), 186–191.

The author discusses the application of the sieve of Eratosthenes to various problems of number theory. He tabulates the primes of the form a^2+1 for $1\leq a\leq 3500$. (Euler made a table for $1\leq a\leq 1500$ [Novi Comment. Acad. Sci. Imp. Petropolitanae **9** (1762–1763), 99–153], and the author corrects a few of the mistakes of Euler's table). He also discusses prime twins and primes of the form x^2+x+a.

P. Erdös (Birmingham).

Cugiani, Marco. Relazione su un gruppo di ricerche di aritmetica additiva dei numeri liberi da potenze. Boll. Un. Mat. Ital. (3) **11** (1956), 359–367.

The problem of the number $E(g, t, N)$ of representations of the positive integer N as a sum of a gth power and a number free of tth powers and generalizations of this problem are given an historical survey. Methods of attack are briefly mentioned. [See also Cugiani, Riv. Mat. Univ. Parma **2** (1951), 403–416; Ann. Mat. Pura Appl. (4) **33** (1952), 135–143; MR **13**, 914; **14**, 356.]

D. H. Lehmer (Berkeley, Calif.).

Cohen, Eckford. The finite Goldbach problem in algebraic number fields. Proc. Amer. Math. Soc. **7** (1956), 500–506.

In an earlier paper the author has shown how elements of the ring of residue classes (mod m) for a rational integer m can be representated as a sum of primes in $R(m)$ under different conditions on m [same Proc. **5** (1954), 478–483; MR **16**, 14]. Here he generalizes his results to the case of a ring of residue classes (mod A), where A is a proper ideal of an algebraic number field F. The problem of representing a number ρ as a sum of primes in $R(A)$ is reduced to the question of solving congruences modulus maximal prime-power divisors of A.

H. Bergström (Göteborg).

Pagni, Plinio. Studio sulle partizioni numeriche. I, II, III. Period. Mat. (4) **32** (1954), 172–183, 199–211, 294–301.

This is an elementary expository paper on the theory of the partitions of an integer, in which the author's aim is to obtain results using methods which are "direct and of easy access". No use is made of generating functions. The paper is in three parts. Part one gives a classification of the types of partitions and recurrence formulas. Part two gives numerical examples and short tables of values of the various types of partitions. Part three is a study of the number of partitions of an integer n into not more than k positive integers, repetitions allowed.

W. H. Simons.

Šidlovskiĭ, A. B. On algebraic independence of transcendental numbers of a certain class. Dokl. Akad. Nauk SSSR (N.S.) **108** (1956), 400–403. (Russian)

The theorem announced by the author in a previous paper [same Dokl. (N.S.) **105** (1955), 35–37; MR **17**, 947] is applied to prove the algebraic independence of each of various sets of hypergeometric E-functions, such as

$$\psi_k(z)=\sum_{n=0}^{\infty}\frac{1}{(n!)^k}\left(\frac{z}{k}\right)^{kn} (k=1, 2, \cdots, r),$$

or of the values of such functions at $z=\alpha$, a nonzero algebraic number.

W. J. LeVeque.

See also: Tsuji, p. 385; Kneser, p. 403.

Theory of Algebraic Numbers

Popovici, Constantin P. La théorie locale des nombres idéaux d'après Zolotarev dans le cas des entiers de Dirichlet. Acad. R. P. Romîne. Stud. Cerc. Mat. **7** (1956), 37–79. (Romanian. Russian and French summaries)

The author makes a detailed study of the divisibility and ideal theory of numbers of the form $\Omega=\alpha+\beta\sqrt{\delta}$, where α, β are complex numbers with rational real and imaginary parts and $\delta=x+iy$, where x, y are rational integers having no common square factor. These numbers are called Dirichlet numbers and Dirichlet integers in case 2α and $\alpha^2-\beta^2\delta$ are Gaussian integers. The introduction of ideals in this relative quadratic field is done according to the theory developed by Zolotarev [J. Math. Pures Appl. (3) **6** (1880), 51–84, 129–166].

D. H. Lehmer (Berkeley, Calif.).

See also: Pyateckiĭ-Šapira, p. 378; Hemer, p. 379; Cohen, p. 382; Šidlovskiĭ, p. 382.

Geometry of Numbers

Schmidt, Wolfgang. Eine Verschärfung des Satzes von Minkowski-Hlawka. Monatsh. Math. **60** (1956), 110–113.

Let S be a bounded Jordan-measurable set not containing the origin. When he gave a proof of a similar statement of Minkowski, E. Hlawka [Math. Z. **49** (1943), 285–312; MR **5**, 201] proved that there will be a lattice with determinant 1, having no point in S, provided the volume of S satisfies $V(S)<1$. The condition $V(S)<1$ has been relaxed by the author [Monatsh. Math. **60** (1956), 1–10; MR **18**, 21] and by the reviewer [Philos. Trans.

Roy. Soc. London. Ser. A. **248** (1955), 225–251; MR **17**, 242]. Here the author proves the statement under the condition $V(S)<2(1+2^{1-n})^{-1}(1+3^{1-n})^{-1}$. For small values of n this is the best result obtained so far. The proof is simple, but is concise and is best read in conjunction with the paper by the author referred to above.

C. A. Rogers (Birmingham).

Tsuji, Masatsugu. Analogue of Blichfeldt's theorem for Fuchsian groups. Comment. Math. Univ. St. Paul. **5** (1956), 17–24.

Let G be a Fuchsian group of linear transformations which leave $|z|<1$ invariant and let D_0 be its fundamental domain, assumed of finite non-euclidean area $\sigma(D_0)$. Let L be the lattice formed by k given points in D_0 and their equivalents by G. Let E be a measurable set contained in $|z|\leqq\rho_0<1$ and let $E(a)=T_aE$ be the transform of E by $T_a{:}z'=(1+z)/(1+\bar{a}z)$ $(|a|<1)$. If $\mu(E(a))$ denotes the number of lattice points contained in $E(a)$, the author proves the following relation

$$\lim_{r\to1}\left[\left(\int_0^r\frac{d\rho}{1-\rho}\int_0^\rho d\lambda\int_{|a|\leqq\lambda}\mu(E(a))d\sigma(a)\right)\left(\log\frac{1}{1-r}\right)^{-2}\right]=\frac{2\pi k\sigma(E)}{\sigma(D_0)},$$

where $d\sigma(a)=4r(1-r^2)^{-2}drd\theta$ $(a=re^{i\theta})$. From this theorem it follows an analogue of Blichfeldt's theorem for translation groups on the euclidean plane. *L. A. Santaló.*

Ehrhart, Eugène. Sur les polygones plans dans un réseau de l'espace. C. R. Acad. Sci. Paris **242** (1956), 1844–1846.

This gives an extension of a result obtained in same C. R. **242** (1956), 1570–1573 [MR **17**, 948]. Here, the polygons are regarded as having vertices at points of a sub-lattice of a lattice in 3-space.

J. H. H. Chalk (London).

Ehrhart, Eugène. Sur les polyèdres et les ovales. C. R. Acad. Sci. Paris **242** (1956), 2217–2219.

The results of the paper reviewed above are used to determine a relation between the number of points in 3-space with integral coordinates, lying on the surface of an arbitrary polyhedron whose vertices also have integral coordinates.

It is also shown that if an oval in the plane contains more than 4 points of a lattice, then at least 3 of them are collinear. *J. H. H. Chalk* (London).

Ehrhart, Eugène. Sur des polygones et des polyèdres particuliers. C. R. Acad. Sci. Paris **243** (1956), 347–349.

In previous notes [same C. R. **242** (1956), 1570–1573; MR **17**, 948; see also the two papers reviewed above] the problem of determining the number of lattice points in a polygon, or on the surface of a polyhedron, has been considered in the case when the vertices are also lattice points. Here, the corresponding problem for the number of lattice points inside, or on the boundary of, a prism is solved.

A result for polygons, without the restriction that the vertices be lattice points, is also given.

J. H. H. Chalk (London).

See also: Šidlovskiĭ, p. 382.

ANALYSIS

Functions of Real Variables

Neĭšuler, L. Ya. k-term tabulation, reduction of the number of experiments in compiling the "Initial table" and splitting of empirical functions of three or four variables. Dokl. Akad. Nauk SSSR (N.S.) **110** (1956), 176–179. (Russian)

In this paper the author touches upon the problem of discovering representations of two forms

$$f(x, y, z)=f_2[f_1(x, y), z]$$

$$f(x, y, z, u)=f_3[f_1(x, y), f_2(z, u)]$$

when the function f is empirical. This paper is a continuation of a long series of papers on this subject. [See MR 14, 504 for references to earlier papers.] The discussion is only general. *D. H. Lehmer* (Berkeley, Calif.).

Trenogin, V. A. On the uniqueness of representation of functions of several variables by superposition of functions of a smaller number of variables in Banach spaces. Dokl. Akad. Nauk SSSR (N.S.) **110** (1956), 184–187. (Russian)

This is a generalization {for the sake of generalization} of a problem of Neĭšuler [see the paper reviewed above] dealing with the unique representation of a function of n variables by means of n functions of two variables each. The generalization consists in replacing functions by operators in Banach spaces and in allowing any number of variables to occur in the representing operators. Conditions for uniqueness are expressed in terms of existence of certain subspaces of the Banach spaces and inverses of certain Fréchet derivatives. *D. H. Lehmer.*

See also: Pozzolo Ferraris, p. 412.

Measure, Integration

Cotlar, Mischa. On the algebraic theory of the mean and the Hahn-Banach theorem. Rev. Un. Mat. Argentina **17** (1955), 9–24 (1956). (Spanish)

The author establishes a general form of the Hahn-Banach theorem, and from it deduces various known results (and extensions thereof) on the existence of invariant means, invariant linear functionals, etc. Let X be a commutative semigroup with zero, $\prec$ a reflexive partial ordering of X which is preserved under addition, and R the real number field. For $Y\subset X$, $e\in Y$, and $a\in R$ (and for p a subadditive function on X to R), let $[Y; e, a]$ ($[Y; e, a; p]$) be the set of all functions f on Y to R such that $f(e)=a$, and $\sum f(x_i)\leqq\sum f(y_j)$ $(\sum f(x_i)\leqq p(z)+\sum f(y_j))$ whenever the x_i's and y_j's (and z) are in Y and $\sum x_i\leqq\sum y_j$ $(\sum x_i\leqq z+\sum y_j)$. (Note that if f is on X to R and $f(e)=a$, then $f\in[X; e, a]$ if and only if f is additive and monotone, and $f\in[X; e, a; p]$ if and only if $f\in[X; e, a]$ and $f\leqq p$. For $e\in X$, let $C'(e)$ $(C(e))$ denote the set of all $x\in X$ for which there exist integers m, n, and $n'\geqq0$ (and points z, $z'\in X$) with $me\prec nx$ and $x\prec n'e$ $(me\prec nx+z$ and $x\prec n'e+z')$. The author's basic theorem is that if $e\in Y\subset C(e)$ and $f_0\in[Y; e, a; p]$, then there is an extension $f\supset f_0$ with $f\in[C(e); e, a; p]$. A corollary asserts that $[C'(e); e, a]$ is nonempty if and only if $ne\prec me$ implies $na\leqq ma$ for all integers $n, m\geqq0$.

Taking $x\prec y$ if there exists $z\in X$ with $x+z=y$, the

corollary yields a result due to Tarski [Fund. Math 31 (1938), 47–66]. With X a real linear space and $x \prec y$ if and only if $x=y$, the author's basic theorem yields the Hahn-Banach theorem. In similar manner, there are obtained the results of von Neumann on invariant means [ibid. 13 (1929), 73–116, 333], and results of Agnew [Duke Math. J. 4 (1938), 55–77] and Agnew and Morse [Ann. of Math. (2) 39 (1938), 20–30] on invariant extension of linear functionals. *V. L. Klee* (Seattle, Wash.).

★**Kamke, E. Das Lebesgue-Stieltjes-Integral.** B. G. Teubner Verlagsgesellschaft, Leipzig, 1956. vi+226 pp. DM 20.00.

This is in a sense a second edition of the author's monograph "Das Lebesguesche Integral" [Teubner, Leipzig-Berlin, 1925], but is completely rewritten and reflects the change in emphasis from the one-dimensional Lebesgue integral to the n-dimensional Lebesgue-Stieltjes integral. The first chapter is devoted to a brief introduction to set theory and the topological properties of sets in n-dimensional space. The second chapter on content and measure develops the theory of measure of n-dimensional sets on the basis of additive functions (called Eichfunktionen) on closed intervals $a_i \leq x_i \leq b_i$ ($i=1, \cdots, n$). The extension to open sets and the definition of upper measure follow in the usual fashion. Measurability of a set E on the basis of an upper measure $\overline{\phi}$ is defined as the existence for every $\varepsilon>0$ of an open set $G_\varepsilon \supset E$ such that $\overline{\phi}(G_\varepsilon - E)<\varepsilon$, leading to the usual properties of measurable sets and measurability. The third chapter discusses Lebesgue-Stieltjes integrals at first for positive functions, based on the measurability of the ordinate set in terms of the product measure $(y_2-y_1)\phi(I)$. The theory is then extended to the integrals of functions of arbitrary sign relative to functions of bounded variation, i.e. the difference of two measure functions. The space L^2 including the Riesz-Fischer theorem as well as an existence theorem for a system of "differential" equations based on Lebesgue integrals closes this chapter. The last chapter gives a brief exposition of the Perron integral in one variable including substitution theorems. The book, like its predecessor, is not only a readable introduction but provides good coverage of the subject in its 222 pages.
T. H. Hildebrandt (Ann Arbor, Mich.).

Klement'ev, Z. I. Compactness of a family of completely additive functions. Tomskiĭ Gos. Univ. Uč. Zap. Mat. Meh. 25 (1955), 9–12. (Russian)

The author proves that a set of measures is compact (relative to the topology of convergence on every set of the countably generated Borel field of definition) if and only if it is bounded and uniformly additive. [See also Dubrovsky, Mat. Sb. N.S. **20(62)** (1947), 317–329; MR **9**, 19; and Bartle, Dunford, and Schwartz, Canad. J. Math. **7** (1955), 289–305; MR **16**, 1123.] *M. M. Day.*

Perkal, J. On the ε-length. Bull. Acad. Polon. Sci. Cl. III. 4 (1956), 399–403.

A modification of the notion of length is proposed to suit the "naturalist". *L. C. Young* (Madison, Wis.).

Young, Laurent C. Champs vectoriels attachés à une mesure plane. J. Math. Pures Appl. (9) **35** (1956), 345–358.

The paper characterizes generalized graphs in a plane X whose boundaries are subject to certain conditions. Let F be the space of all continuous functions $f(x, j)$ of a point x of X and a vector j which are positively homogeneous in j, F_K the space of all f of F linear in j, and F_H the space of all exact f of F; then $F_H \subseteq F_K$. A (generalized) graph L is defined as a linear and positive functional on F. In particular, ordinary rectifiable or polygonal curves are graphs, and so are the generalized curves defined previously by the author, namely weak limits of sequences of polygonal curves. The track C of L is its restriction to F_K, and can be represented by a positive measure μ with a compact support and a vectorfunction $j(x)$ such that $[C, f]=\int_X f(x, j(x))d\mu(x)$. The restriction $bL=bC$ of L or C to F_H is termed the boundary of L and C, and L and C are called closed if $bL=0$. A graph L_0 representable as $L_0(f)=\int_E L(f)d\alpha(L)$ where α is a positive and finite measure on a set E of graphs, is said to be a mixture of E with weight $\alpha(E)$; likewise for tracks or boundaries. Let Λ' be the space of all boundaries of polygonal graphs, i.e. linear combinations with positive coefficients of polygonal curves, and Λ the closure of Λ' in a topology which is essentially based on the norm "$\|\lambda\|$=infimum of all $\mu(X)$ with $bC=\lambda$". Then any L with $\lambda_0=bL \in \Lambda'$ is the weak limit of a sequence of polygonal graphs with the same boundary, the closed graphs are the mixtures of closed generalized curves, and if $\lambda_0=bL\neq 0$ is the boundary of some polygonal curve, L is the sum of a closed graph and a mixture with weight 1 of generalized curves with boundary λ_0. We have $bC \in \Lambda$ if and only if C is a mixture of tracks of rectifiable curves. The proofs are founded on a variational homology principle, and applications to the calculus of variations are mentioned. *K. Krickeberg.*

Špaček, Antonín. Zufällige Mengenfunktionen. Math. Nachr. **14** (1955), 355–360 (1956).

Let $(X, \mathfrak{S}, \nu)$ be a probability space, and let F be the set of all real-valued functions on $\mathfrak{S}$. Let $\mathfrak{F}$ be the least σ-algebra of subsets of F such that for each A in $\mathfrak{S}$ the function $\varphi \rightarrow \varphi(A)$ on F is measurable with respect to $\mathfrak{F}$. If μ is a probability measure on $\mathfrak{F}$, then $(F, \mathfrak{F}, \mu)$ is called a random setfunction. The author studies conditions under which this random setfunction is absolutely continuous, i.e., the subset W of F consisting of the probability measures on $\mathfrak{S}$ that are absolutely continuous with respect to ν has outer measure 1 with respect to μ. In case the random measure is absolutely continuous, the author defines its Radon-Nikodym derivative and proves that in a special case (Borel measures in separable metric spaces) the derivative is a random function, i.e., for each fixed φ in W it is a measurable function on X.
P. R. Halmos (Chicago, Ill.).

See also: Smith, p. 389; Ivanov, p. 389; Hlawka, p. 390; Kneser, p. 403; Cristéscu, p. 404.

Functions of Complex Variables

Lambin, N. V. Poles of Σ-monogenic functions. Belorussk. Gos. Univ. Uč. Zap. Ser. Fiz.-Mat. **15** (1953), 7–13. (Russian)

The author describes the topological character of poles of sigma-monogenic functions [L. Bers and A. Gelbart, Trans. Amer. Math. Soc. **56** (1944), 67–93; MR **6**, 86]. The result is included in the more general theory of pseudoanalytic functions [L. Bers, Proc. Nat. Acad. Sci. U.S.A. **36** (1950), 130–136; MR **12**, 173]. *L. Bers.*

Ricci, Giovanni. Complementi a un teorema di H. Bohr riguardante le serie di potenze. Rev. Un. Mat. Argentina **17** (1955), 185–195 (1956).

The author denotes by $\bar{F}_h$ the class of power series $f(z)=\sum_{n=h}^{\infty} a_n z^n$ beginning with terms in z^h, converging in $|z|<1$ and bounded there by 1. Let $M(r)$ denote the maximum modulus and $\mathbf{M}(r)$ the majorant $\sum |a_n| r^n$. The theorem of the title states that if $f\in \bar{F}_0$ then $\mathbf{M}(\frac{1}{3})\leq 1$ and no larger number than $\frac{1}{3}$ will do. Let $B_h=\sup \gamma$ such that $\mathbf{M}(\gamma)\leq 1$ for $f\in \bar{F}_h$, so that $\frac{1}{3}=B_0\leq B_1\leq\cdots\leq 1$. The author obtains the following information about B_h. First, $B_h\uparrow 1$ as $h\uparrow\infty$, and $B_h+B_h{}^h>1$. Furthermore, let $F_h(\alpha)$ denote the subset of $\bar{F}_h$ for which $|a_h|=\alpha$, and define $B_h(\alpha)$ for $F_h(\alpha)$ as B_h is defined for $\bar{F}_h$. Then $1/(2+\alpha)\leq B_0(\alpha)\leq 1/(1+2\alpha)$ and $B_h(\alpha)\to 1$ as $\alpha\to 1$. Upper and lower bounds are given for $B_h(\alpha)$ (roots of certain polynomials), and a lower bound for B_h.

R. P. Boas, Jr. (Evanston, Ill.).

Gaier, Dieter; und Meyer-König, Werner. Singuläre Radien bei Potenzreihen. Jber. Deutsch. Math. Verein. **59** (1956), Abt. 1, 36–48.

Let (1) $\sum a_n z^n=f(z)$ be regular and unbounded in the unit disk D; and let a point $e^{i\theta}$ be called B-singular or B-regular according as f is or is not unbounded in every open sector of D which contains the radius of $e^{i\theta}$. For every series (1), there exists a sequence ε_n ($\varepsilon_n=\pm 1$) such that each point $e^{i\theta}$ is B-singular for $\sum \varepsilon_n a_n z^n$. If all the coefficients in (1) lie in a sector of the plane whose angle is less than π, then $z=1$ is B-singular. If (1) has Ostrowski gaps, then the partial sums of the series (1) whose indices fall into the gaps are uniformly bounded on every closed arc of B-regular points; moreover, at each B-regular point at which the radial limit $f(e^{i\theta})$ exists, these partial sums converge to $f(e^{i\theta})$. If (1) has Hadamard gaps, all points $e^{i\theta}$ are B-singular. It remains an open question whether the analogue to Fabry's gap theorem is true. Since the function $f(z)=\sum (1-z)^n z^{n!}$ is not rational and is B-regular at $z=1$, the analogue to the Pólya-Carlson theorem on series with integral coefficients is false. Likewise, the authors show that the analogue to Hadamard's theorem on the composition of singularities is false.

Roughly speaking, almost all functions $f(z)$ have no B-regular points on $|z|=1$. In this connection, the authors establish a proposition which is strictly analogous to Bagemihl's recent theorem on strong and weak points [Michigan Math. J. **3** (1955–56), 133–135; MR **17**, 1195].

G. Piranian (Ann Arbor, Mich.).

Skof, Fulvia. Osservazioni sulle componenti lacunari delle serie ultraconvergenti. Boll. Un. Mat. Ital. (3) **11** (1956), 217–228.

Let (1) $\sum a_n z^n=f(z)$ be regular in $|z|<1$ and on some arc of $|z|=1$, and let the series have order Λ of H-O (Hadamard-Ostrowski) lacunarity as defined by G. Ricci [Rend. Math. e Appl. (5) **14** (1955), 602–632; MR **17**, 598]. Let (2) $\sum a_{n_k} z^{n_k}=g(z)$ be a lacunary component of (1); that is, let $\limsup n_{k+1}/n_k>1$, and let $f(z)-g(z)$ be regular on $|z|\leq 1$. The author investigates the function

$$S(x)=\sum_{n_k\leq x} 1/n_k$$

associated with the logarithmic density of the index sequence $\{n_k\}$.

For every lacunary component (2), the function $S(x)$ is unbounded. If $\Lambda<\infty$, there exists a positive constant $\gamma=\gamma(g)$ such that (3) $S(x)>\gamma \log x$ ($x>x_0$). For the case $\Lambda=\infty$, the conclusion no longer holds; there exist functions (1) with $\Lambda=\infty$ for which (3) holds for every lacunary component; and there exist other functions (1) with lacunary components for which $S(x)=o(\log x)$.

If $\Lambda=\infty$ and (2) satisfies certain conditions which can be described roughly by saying that not very far before and after each H-O gap there exist coefficients which are not too small, then $S(x)$ is minimal in the following sense: If the lacunary component g^* is contained in and adherent to g; that is, if the power series of g^* is a subseries of (2) and if each of its H-O gaps contains precisely one of the H-O gaps of (2); then there exists a positive constant $\gamma=\gamma(g, g^*)$ such that $\gamma S(x)\leq S^*(x)$.

Suppose that $\psi(x)\nearrow\infty$ as $x\to\infty$, and that $\psi'(x)=O(1/x)$. Then there exists a series (1) with a lacunary component (2) such that

$$\gamma\psi(x)\leq S(x)\leq\psi(x)\quad (\gamma=\gamma(g)>0),$$

and such that, for every lacunary component g^* which is contained in g and adherent to g,

$$\gamma^*\psi(x)\leq S^*(x)\quad (\gamma^*=\gamma^*(g, g^*)>0,\ x>x_0).$$

G. Piranian (Ann Arbor, Mich.).

Parodi, Maurice. Remarque relative à la localisation des zéros d'un polynome dans le plan complexe. C. R. Acad. Sci. Paris **242** (1956), 2272–2273.

In same C. R. **239** (1954), 1177–1178 [MR **16**, 469] Parodi has shown that the zeros of the polynomial

$$f(z)=z^n+a_1z^{n-1}+\cdots+a_n\quad (a_n\neq 0),$$

lie in the domain $D=C_1+C_2$, where C_1 and C_2 are the interiors of the circles $|z|=1$ and $|z+a_1|=|\sum_{k=2}^{n} a_k|$ respectively. On considering the polynomial symmetric to $f(z)$ relative to the unit circle, Parodi concludes in the present paper that all the zeros of $f(z)$ lie also in the domain consisting of the union of C_1 with C_3, the circle obtained by inverting in the unit circle the circle

$$|z+(\bar{a}_{n-1}/\bar{a}_n)|\leq|a_n|^{-1}(1+|a_1|+\cdots+|a_{n-2}|).$$

M. Marden (Milwaukee, Wis.).

Uluçay, Cengiz. Note on Schiffer's variation formula. Arch. Math. **7** (1956), 291–294.

Hadamard's variational formula for the Dirichlet Green's function [M. Schiffer, Bull. Amer. Math. Soc. **60** (1954), 303–328; MR **16**, 233] is proved under weak hypotheses on the boundary of the region G, under the assumption that the varied region G^* is obtained by the mapping

$$z^*=z+\frac{e^{2i\theta}\rho^2}{z-z_0}\quad (\rho>0,\ 0\leq\phi\leq\pi),$$

z_0 being an interior point of G [cf. Schiffer, p. 307].

R. B. Davis (Syracuse, N.Y.).

Nikolaeva, G. A. On approximate conformal mapping by means of conjugate trigonometric series. Dokl. Akad. Nauk SSSR (N.S.) **110** (1956), 180–183. (Russian)

The author considers a method of L. V. Kantorovic [Mat. Sb. **40** (1933), 294–325] for the approximate conformal mapping of a circle into a region with boundaries given analytically. After describing some variations of the method the author presents a new consideration of the problem of the convergence and the estimation of the

accuracy of the method by transforming the problem to the solution of a non-linear functional equation.
S. Kulik (Columbia, S.C.).

Arima, Kihachiro. On the modulus of integral function. Sci. Rep. Saitama Univ. Ser. A. 2 (1956), 87–93.

The author extends a theorem of F. Wolf [J. London Math. Soc. 14 (1939), 208–216; Bull. Amer. Math. Soc. 48 (1942), 925–932; MR 1, 48; 4, 144]. His first theorem is as follows. Let D be a region whose boundary meets each circle $|z|=r$ for sufficiently large r. If $u(z)$ is subharmonic in D, with $u(z)\leq 0$ on the boundary and $|u(z)|\leq r^{\frac{1}{2}}e^{\phi(\theta)}$ in D, where $\phi(\theta)$ is nonnegative and Lebesgue integrable, then $r^{-\frac{1}{2}}u(z)$ is bounded in D. Hence if $u(z)$ is subharmonic everywhere, if $m(r)=\inf_{|z|=r}u(z)$ is bounded above, and $u(z)\leq r^{\frac{1}{2}}e^{\phi(\theta)}$, it follows that $r^{-\frac{1}{2}}u(z)$ is bounded. Hence, further, if $f(z)$ is an entire function and its minimum modulus is bounded, and $|f(z)|\leq\exp(r^{\frac{1}{2}}\phi(\theta))$ then $|f(z)|\leq\exp(Kr^{\frac{1}{2}})$. If $|f(z)|\leq\exp(\varepsilon r^{\frac{1}{2}}\phi(\theta))$ for arbitrarily small positive ε and sufficiently large r, then $f(z)$ is a constant. Further theorems have $r^{1/p}$ instead of $r^{\frac{1}{2}}$ with more restrictions on the size of D. {For an extension of Wolf's theorem in another direction see A. W. McMillan, Amer. J. Math. 66 (1944), 405–410; MR 6, 61, 334. It should be possible to combine the two directions of generalization.} *R. P. Boas, Jr.* (Evanston, Ill.).

Hervé, Michel. Contribution à l'étude d'une fonction méromorphe au voisinage d'un ensemble singulier de capacité nulle. J. Math. Pures Appl. (9) 35 (1956), 161–173.

Let D be an arbitrary domain, C the boundary of D, E a closed set of capacity zero contained in C and $f(z)$ a single-valued meromorphic function in D. The cluster sets $S_\zeta^{(\Delta)}$ and $S_\zeta^{*(C)}$ of $f(z)$ at $\zeta\in C$ are defined as follows: $w\in S_\zeta^{(\Delta)}$ if there exists a sequence of points z_n in Δ such that $\zeta=\lim z_n$ and $w=\lim f(z_n)$, where Δ is an arbitrary set such that $\Delta\subset D$ and $\zeta\in\overline{\Delta}$; $w\in S_\zeta^{*(C)}$ if there exist a sequence of values w_n and a sequence of points ζ_n of $C-\zeta-E$ such that $w_n\in S_{\zeta_n}^{(D)}$, $\zeta=\lim\zeta_n$ and $w=\lim w_n$. If $f(z)\neq w$ in the intersection of D with a neighborhood of ζ, w is called an exceptional value of $f(z)$ at ζ. Suppose that $C-C'$ and $C-\overline{(C-E)}$ consist of only essential singularities of $f(z)$. Then, the set $\Omega=S_\zeta^{(D)}-S_\zeta^{*(C)}$ is an open set. The author has proved that if each point of E sufficiently near ζ belongs to a non-degenerate continuum disjoint with D, then each connected component of Ω contains at most two exceptional values of $f(z)$ at ζ [C. R. Acad. Sci. Paris 240 (1955), 718–720; MR 16, 684]. The purpose of the present paper is to investigate properties of the set E and the cluster sets $S^{(\Delta)}$ which can be deduced from the assumption that a connected component of Ω contains at least three exceptional values of $f(z)$ at ζ.

First, the author proves: If $\zeta\in E$, if a connected component of $\Omega=S_\zeta^{(D)}-S^{\zeta*(C)}$ contains three exceptional values w, w', w^* of $f(z)$ at ζ, and if U is a domain containing these three values such that $\overline{U}\subset\Omega$, then there exists a perfect subset P of E, containing ζ, with the following properties: If $\zeta'\in P$, then $U\subset S_{\zeta'}^{(D)}-S_{\zeta'}^{*(C)}$; if $\zeta'\in C-P$ and $|\zeta'-\zeta|$ is sufficiently small, then $U\cap S_{\zeta'}^{(D)}=0$. From this theorem follows that if the intersection of E with a neighborhood of ζ is countable, then every connected component of Ω contains at most two exceptional values of $f(z)$ at ζ.

Next, the author states that under the same assumption as in the above theorem, there exist a positive number α and a subset p_1 of the first category on P (defined above) with the following properties: If $\xi\in P$ and if Δ is a simple curve in D terminating at ξ such that the diameter of $S_\xi^{(\Delta)}$ is less than α, then $\xi\in p_1$. In particular, all the points of P where $f(z)$ admits asymptotic values belong to p_1. Using an interesting lemma, the author proves a more general theorem. For this purpose, the author defines a subset $p(m, X)$ of P in the following way: $\xi\in p(m, X)$ if there exists a continuum Ξ such that (a) $\xi\in\Xi$; (b) the diameter of $\Xi\geq 1/m$ (m: entire); (c) $z\in\Xi-C$ implies $z\in D$ and $f(z)\in X$. The author proves: Under the same assumption as above, the set $p(m, X)$ is nowhere dense on P for every closed set X such that a connected component of $U-X$ contains two exceptional values w' and w^* of $f(z)$ at ζ. Finally the author proves: If $w\in S_\zeta^{(D)}-S_\zeta^{*(C)}$, if $w=\lim w_p$ and if $f(z)\neq w_p$ for $z\in D$ and $|z-\zeta|<r$ for all p, then all the points of E where $f(z)$ admits w as an asymptotic value form a set which contains a perfect set.
K. Noshiro (Nagoya).

Tsuji, Masatsugu. Borel's direction of a meromorphic function in a unit circle. J. Math. Soc. Japan 7 (1955), 290–311.

The author establishes the following theorem. Let $f(z)$ be a meromorphic function of finite order $\rho>0$ in $|z|<1$. Then there exists a point z_0 on $|z|=1$ and a line J through z_0, which is directed into the interior of $|z|=1$, but may also coincide with the tangent to $|z|=1$ at z_0, satisfying the following condition: If ω denotes any angular domain containing J and bounded by two lines through z_0, $g(z)$ is any function meromorphic in $|z|<1$ and of order $<\rho$ there, and $z_\nu(f=g, \omega)$ are the zero of $f(z)-g(z)$ in ω, multiple zeros being counted once only, then for any $\varepsilon>0$

$$\sum_\nu\{1-|z_\nu(f=g, \omega)|\}^{\rho+1-\varepsilon}=\infty,$$

except possibly for two functions $g(z)$. If, in addition, $f(z)$ is of divergence type and $g(z)$ is such that

$$\int_0^1 T(r, g)(1-r)^{\rho-1}<\infty,$$

then

$$\sum_\nu\{1-|z_\nu(f=g, \omega)|\}^{\rho+1}=\infty,$$

except possibly for two functions $g(z)$.

This result represents an analogue for $|z|<1$ of two earlier theorems proved for $|z|<\infty$ by M. Biernacki [Acta Math. 56 (1930), 197–204] and A. Rauch [J. Math. Pures Appl. (9) 12 (1933), 109–171]. [See also M. Tsuji, Tôhoku Math. J. (2) 2 (1950), 97–112; Kôdai Math. Sem. Rep. 1950, 104–108; MR 12, 815, 816.] *W. Seidel.*

Geronimus, Ya. L. On some properties of analytic functions continuous on a closed circle or circular sector. Mat. Sb. N.S. 38(80) (1956), 319–330. (Russian)

Among the results of this paper, the following are typical. If $f(z)$ is regular in $|z|<1$ and continuous in $|z|\leq 1$ with modulus of continuity $\omega(\delta)$ on $|z|=1$, then in $|z|<1$,

$$|f'(re^{i\varphi})|\leq C\omega\left[(1-r)\log\frac{b}{1-r}\right]/(1-r)\quad(r<1, b>1),$$

and for $|z_1|\leq 1$, $|z_2|\leq 1$,

$$|f(z_1)-f(z_2)|\geq C'\omega\left[|z_1-z_2|\log\frac{b'}{|z_1-z_2|}\right]\quad(b'>2).$$

Similarly, let $f(z)$ be regular in $|z|<1$ and satisfy there the inequality $|f(re^{i\varphi})|\leqq\psi(1-r)$ $(r<1)$, where $\psi(x)$ tends to infinity with $1/x$ in such a way that $\int_0^\eta \psi(x)dx<\infty$. Then the primitive $F(z)=\int f(z)dz$ is continuous in $|z|\leqq 1$ with modulus of continuity $\omega(\delta)$ on $|z|=1$ satisfying the inequality $\omega(\delta)\leqq C\lambda(\delta)$, where $\lambda(x)=\int_0^x \psi(x)dx$. Furthermore, $|F(z_1)-F(z_2)|\leqq C'\lambda(|z_1-z_2|)$ for $|z_1|\leqq 1$, $|z_2|\leqq 1$. As pointed out by the author, these theorems generalize certain earlier results of Hardy and Littlewood [Math. Z. **34** (1932), 403–439] and those of Gagua [Soobšč. Akad. Nauk Gruzin. SSR **10** (1949), 451–456; MR **14**, 547]. These results are modified to the case that the functions are assumed to be continuous not on the whole circumference $|z|=1$, but only on an arc. Making use of a theorem due to Walsh and Elliott [Proc. Nat. Acad. Sci. U.S.A. **38** (1952), 1058–1066, p. 1060; MR **14**, 741] on conjugate functions, theorems on the correspondence of frontiers under conformal mapping are obtained both in the global and local form. Finally, similar theorems are obtained for functions of class H_p, $p>1$. *W. Seidel.*

Suvorov, G. D. On the continuity in the closed circle of functions regular in the open circle. Uspehi Mat. Nauk (N.S.) 11 (1956), no. 3(69), 177–179. (Russian)

The author discusses what is essentially Theorem 13 of his earlier paper on prime ends of sequences of domains [Mat. Sb. N.S. **33(75)** (1953), 73–100; MR **15**, 244; **17**, 472]. *G. Piranian* (Ann Arbor, Mich.).

Jenkins, James A. On a problem of Lusin. Michigan Math. J. 3 (1955–1956), 187–189.

By using power series with Hadamard gaps, Lohwater and the reviewer have constructed a function which is bounded and regular in the unit disk D and which has every point $e^{i\theta}$ as a Lusin point [Michigan Math. J. **3** (1955–56), 63–68; MR **17**, 834]. The author approaches the same problem by constructing an appropriate simply connected Riemann surface R and considering an arbitrary conformal mapping f of D onto R. For the construction of R, he uses denumerably many copies of the unit disk, each with infinitely many radial slits, and he cross-joins the slit disks pairwise, in such a way that all the copies lie above the disk $|w|<1$.

While it is highly plausible that with this construction every point $e^{i\theta}$ is a Lusin point of f, the author claims and proves a little less, namely that the Lusin points of f constitute a residual set of measure 2π. The interest of the paper lies in the simplicity of the example and in the fact that the radial limit of f has modulus 1, wherever it exists. *G. Piranian* (Ann Arbor, Mich.).

Biernacki, Mieczysław. Sur les fonctions en aire multivalentes. Ann. Univ. Mariae Curie-Skłodowska. Sect. A. 8 (1954), 71–79 (1956). (Polish and Russian summaries)

Let $f(z)=z^p+\sum_{n=p+1}^\infty a_nz^n$ be regular and area mean p-valent in $|z|<1$. Let $M(r)=\max|f(re^{i\theta})|$,

$$\mathfrak{M}(r)=r^p+\sum_{n=p+1}^{\infty}|a_n|r^n$$

and set $zf'(z)/f(z)=p+\sum_{n=1}^\infty b_nz^n$. The author proves: (I) $\liminf_{n\to\infty}|b_n|\leqq B(p)$, but $\limsup|b_n|$ may be infinite; (II) if there is a k such that

$$k^2|a_n|^2\geqq(1+|a_{p+1}|^2+\cdots+|a_{n-1}|^2)/n$$

for $n=1, 2, \cdots$, then $\mathfrak{M}(r)\leqq M(r)C(p)k$; (III) If

$$\varphi_n(z)=\int\varphi_{n-1}(z)dz\ (n=1, 2, \cdots),$$

$\varphi_0(z)=f(z)$, then each of the functions $\varphi_n(z)$, $n\geqq 1$, is the quotient of functions regular and bounded in $|z|<1$; (IV) If $z=re^{i\theta}$, $0<r<1$, then

$$\int_0^{2\pi}\left|\frac{zf'(z)}{f(z)}\right|^2d\theta<\frac{A_1(p)}{1-r}\log\frac{A_2(p)}{1-r}.$$

A. W. Goodman (Princeton, N.J.).

Newns, W. F. Products of basic sets of analytic functions. Proc. Roy. Soc. London. Ser. A. 237 (1956), 55–62.

This is a continuation of an earlier paper by the author [Philos. Trans. Roy. Soc. London. Ser. A **245** (1953), 429–468; MR **14**, 968]. Let $\{p_n^{(1)}\}$ and $\{p_n^{(2)}\}$ be basic sets with $p_n^{(j)}(z)=\sum p_{nk}^{(j)}z^k$, $z^k=\sum\pi_{kn}^{(j)}p_n^{(j)}(z)$. The "product" of these is the set $\{p_n\}$ defined by

$$p_n(z)=\sum p_{nk}^{(1)}p_k^{(2)}(z)$$

with $\pi=\pi^{(2)}\pi^{(1)}$. Results are known which guarantee that if $\{p_n^{(1)}\}$ and $\{p_n^{(2)}\}$ are effective for expanding (in "basic" series) a specific class of analytic functions, then the same holds for the product set. In this note, the author explores a converse problem: assuming that $\{p_n^{(2)}\}$ and $\{p_n\}$ are effective in a class, what can be said about the set $\{p_n^{(1)}\}$? The results are couched in the terminology of his earlier paper, and not easily re-stated. He illustrates the scope and limitations of the results by several effective examples. *R. C. Buck* (Madison, Wis.).

Suetin, P. K. On the representation of analytical functions by series in orthogonal polynomials. Dokl. Akad. Nauk SSSR (N.S.) **109** (1956), 36–39. (Russian)

The author proves two theorems, both depending on results in an earlier paper [same Dokl. (N.S.) **106** (1956), 788–791; MR **18**, 33]. (I) If

$$\int_\Gamma n(\zeta)P_n(\zeta)P_m(\zeta)d\sigma=\begin{cases}0\ (n\neq m),\\1\ (n=m),\end{cases}$$

where Γ, the boundary of a domain G, is a regular analytic curve, $d\sigma$ an element of its arc and $n(z)$ a positive function whose pth derivative exists and satisfies a Lipschitz condition of order $\alpha<1$; then every function $f(z)$ for which the pth primitive is analytic in G can be expended in a series $\sum c_nP_n(z)$ which is uniformly convergent in G. (II) If $z=\psi(w)$ is the inverse of the function which maps G on the unit circle and Γ is a Jordan curve such that $\ln|\psi'(w)|$ is represented by a Poisson integral and $\ln\{n[\psi(w)]\}$ is summable on $|w|=1$, then every function analytic in G can be expanded in a series $\sum c_nP_n(z)$ which is uniformly convergent on any closed set $F\in G$. *S. Macintyre* (Aberdeen).

Bremermann, H. J. Note on plurisubharmonic and Hartogs functions. Proc. Amer. Math. Soc. 7 (1956), 771–775.

On établit a) qu'une fonction $V(z_1, \cdots, z_n)$ plurisousharmonique dans un domaine D de C^n n'est pas nécessairement une fonction de Hartogs [au sens de l'ouvrage de Bochner et Martin, Several complex variables, Princeton, 1948, p. 143; MR **10**, 366] quand D n'est pas un domaine d'holomorphie, et b) que V ne possède pas nécessairement un prolongement plurisousharmonique dans l'enveloppe d'holomorphie $H(D)$. L'exemple d'une fonction convexe $f(x_1, \cdots, x_n)$ de (x), où $x_k=\log|z_k|$, considéré dans un domaine R univalent et cerclé par rapport à l'origine (c'est à dire invariant par les transformations $z_k'=z_ke^{i\theta_k}$ $(1\leqq k\leqq n)$ suffit, R étant pris de manière à avoir dans l'espace des (x) une image d non convexe; il est

bien connu en effect qu'une fonction $f(x)$ convexe dans d n'a pas nécessairement un prolongement convexe dans l'enveloppe de convexité de d. Le mémoire diffère peu d'un précédent [Math. Ann. **131** (1956), 76–86; MR **17**, 1070]; on considère ici des domaines cerclés au lieu de tubes de base d. Le fait qu'une fonction plurisousharmonique des z_k, dépendante des seuls x_k dans R est une fonction convexe quelconque des x_k, établi par le réf. [Ann. Sci. Ecole Norm. Sup. (3) **62** (1945), 301–338, p. 331; MR **8**, 271], entraîne comme conséquence immédiate qu'une fonction $f(x)=f(\log|z_k|)$, plurisousharmonique dans R est une enveloppe supérieure de fonctions $\log|F(z_k)|$, F étant holomorphe dans R, quand on suppose convexe l'image d de R dans l'espace des (x); on fait remarquer ici à juste titre que le réferent avait omis la dernière condition en formulant cet énoncé dans le mémoire cité. *P. Lelong* (Paris).

See also: Sikkema, p. 391; Wintner, p. 393; Bergman, p. 398; Kliot-Dašinskiĭ, p. 420; Lambin, p. 437; Kučerenko, p. 441; Epstein, p. 442; Sconzo, p. 447.

Geometrical Analysis

Obuhov, A. M. Probabilistic description of continuous fields. Ukrain. Mat. Ž. **6** (1954), 37–42. (Russian)

It is well-known that an isotropic tensor $\overline{u_iu_j'}$ (where u_i and u_j' are the values of the field quantities at two points, P and P', say) solenoidal in its indices can be derived in terms of a single scalar Q in the manner

$$\overline{u_iu_j'}=\operatorname{curl} Q(r)\epsilon_{ijk}\xi_k,$$

where $\boldsymbol{\xi}$ is the vector joining PP'; thus

$$(1)\qquad \overline{u_iu_j'}=\frac{Q'}{r}\xi_i\xi_j-(rQ'+2Q)\delta_{ij},$$

where primes attached to scalars such as Q denote differentiations with respect to $r=|\xi|$. Similarly, the tensor $\overline{v_iv_j'}$ where $\mathbf{v}=\operatorname{grad}\phi$ is an irrotational vector can be derived in terms of the scalar $S(r)=\overline{\phi\phi'}$ thus:

$$(2)\qquad \overline{v_iv_j'}=-\frac{\partial^2 S}{\partial\xi_i\partial\xi_j}=\left(\frac{S'}{r}-\frac{S''}{r}\right)\xi_i\xi_j-\frac{S'}{r}\delta_{ij}.$$

If we denote by F_s and G_s and F_n and G_n the corresponding transverse and longitudinal correlations, then

$$(3)\quad F_s=-(rQ'+2Q),\ G_s=-2Q;\ F_n=-S'/r,\ G_n=-S''.$$

From these relations it follows:

$$(4)\qquad rG_s'=2(F_s-G_s) \text{ and } rF_n'=-(F_n-G_n).$$

Now an arbitrary vector field can be expressed as the superposition of a solenoidal ($\mathbf{u}$) and an irrotational ($\mathbf{v}$) vector field. Also the cross correlation $\overline{u_iv_j'}$ is seen to be zero since this is the derivative with respect to r of the solenoidal isotropic vector $\overline{u_i\phi'}$; and it is known that the latter is zero. Thus if $\mathbf{w}=\mathbf{u}+\mathbf{v}$ then $\overline{w_iw_j'}=\overline{u_iu_j'}+\overline{v_iv_j'}$. Thus the two point correlation in an arbitrary vector field can be expressed in terms of two scalars Q and S and is the sum of equations (1) and (2). If $F=(F_s+G_s)$ and $G=(G_s+G_n)$ are the transverse and the longitudinal correlations of this general field then the parts referring to the solenoidal and the irrotational fields can be deduced from the identities:

$$r(G_s'-2F_n')=2(F-G),$$
$$r(G_s'+F_n')+3(G_s+F_n)=2F+G.$$

Thus

$$G_s-2F_n=-2\int_r^\infty(F-G)\frac{d\rho}{\rho},$$
$$G_s+F_n=\int_0^r\rho^2(2F+G)d\rho.$$

Some general comments on Kolmogoroff's similarity principles are also made. *S. Chandrasekhar.*

Imšeneckaya, E. F. On a saltus formula for an integral encountered in the theory of elasticity. L'vov. Politehn. Inst. Nauč. Zap. **30**, Ser. Fiz.-Mat. No. 1 (1955), 15–23. (Russian)

The author considers a certain surface integral in the theory of elasticity [see I. Fredholm, Ark. Mat. Astr. Fys. **2** (1906), no. 28; Acta Math. **23** (1900), 1–42] and arrives at a formula for the "jump" of the integral across the surface S. The integral studied may be regarded as an analogue of the potential of a double layer in the theory of Laplace's equation. As the author points out, the result obtained is not in agreement with the statements in V. D. Kupradze [Boundary problems in the theory of vibrations and integral equations, Gostehizdat, Moscow, 1950, Ch. 4, Sect. 34; MR **15**, 318].

J. B. Diaz (Cambridge, Mass.).

See also: Mauersberger, p. 388; Postnikov, p. 409; Al'ber, p. 409; Miron, p. 413; Varga, p. 414; Guggenheimer, p. 415; Il'yušin, p. 431; Frederick, p. 434; Słowikowski, p. 447.

Functions with Particular Properties

Yeh, G. C. K.; Martinek, J.; and Ludford, G. S. S. A general sphere theorem for hydrodynamics, heat, magnetism, and electrostatics. Z. Angew. Math. Mech. **36** (1956), 111–116. (German, French and Russian summaries)

Aus den kugelsymmetrischen Potentialen $\phi_0(r)$ bzw. $\phi_1(r)$, die Ladungen entsprechen, die außerhalb bzw. innerhalb einer Kugel vom Radius a liegen, wird ein im ganzen Raum erklärtes Potential $\phi(r)$ aufgebaut und dessen Eigenschaften, insbesondere bei Annäherung an die Kugel $r=a$, untersucht. Es zeigt sich, daß sich verschiedene Aufgaben der Hydrodynamik, der Wärmelehre, der Magneto- und Elektrostatik auf Spezialfälle dieses Potentials zurückführen lassen. *K. Maruhn* (Dresden).

Mauersberger, Peter. Über die Unstetigkeiten der zweiten Ableitung des Schwerepotentials an Diskontinuitätsflächen der Dichte. Z. Angew. Math. Mech. **36** (1956), 398.

This paper contains a new discussion of a relation credited to H. Ertel [same Z. **25/27** (1947), 186–189; MR **9**, 284]. It is shown by vectorial methods that if κ is the gravitational constant, $\{\sigma\}$ the jump in the density across a surface for which $\mathfrak{N}$ is the unit normal vector, and if $\mathfrak{u}$ and $\mathfrak{v}$ are arbitrary vectors, then the jump $\{\partial^2\Phi/\partial\mathfrak{u}\partial\mathfrak{v}\}$ in the second derivative of the gravitational potential Φ in the directions of $\mathfrak{u}$ and $\mathfrak{v}$ is given by the formula

$$\left\{\frac{\partial^2\Phi}{\partial\mathfrak{u}\partial\mathfrak{v}}\right\}=4\pi\kappa\{\sigma\}\cdot\cos\sphericalangle(\mathfrak{N},\mathfrak{u})\cdot\cos\sphericalangle(\mathfrak{N},\mathfrak{v}).$$

F. W. Perkins (Hanover, N.H.).

Du Plessis, N. Spherical Fejér-Riesz theorems. J. London Math. Soc. **31** (1956), 386–391.

L'A. considère une fonction harmonique f dans le domaine sphérique de rayon 1 de l'espace euclidien à $n+1$ dimensions. Il suppose que la moyenne sur les sphères concentriques ($OM=\rho<1$) est bornée. On sait que si $r>1$ il y a une limité radiale p.p. notée $f(Q)$ et que sa moyenne est finie, c'est à dire: $M_r{}^r(f)=\int|f(Q)|^r dS<+\infty$. Alors l'A. montre que

$$\int_D(1-\rho)^{m-2}|f(P)|^r dV\leq A_{r,m}M_r{}^r(f)$$

($\rho=OP$, D intersection du domaine avec un hyperplan diamétral de dimension $(n-m+2)$, $A_{r,n}$ fonction de r et n seuls). Adaptation pour le cas $r=1$.

L'A, utilisant une transformation de Kelvin, admet que la fonction harmonique transformée, si elle a une limite le long d'un arc de circle orthogonal à l'hyperplan-frontière, admet la même limite selon la normal au plan. C'est exact au moins si $f\geq 0$ et on peut se ramener à ce cas (parce que f est différence de deux fonctions harmoniques ≥ 0 et que la limite n'intervient que p.p.). *M. Brelot.*

Smith, Kennan T. Mean values and continuity of Riesz potentials. Comm. Pure Appl. Math. **9** (1956), 569–576.

Dans une boule ouverte S de R^n, on considère les fonctions nulles hors d'un compact et $2m$ fois continûment différentiables. Lorsqu'on prend comme carré d'une norme la somme des intégrales des carrés des dérivées d'ordre $\leq m$, on obtient par complétion un espace $P_0{}^m(S)$ dont les éléments sont égaux à des potentiels de M. Riesz $\int K_m(x-y)g(y)dy$ (où K_m est proportionnel à $|x|^{m-n}$ sauf sur un ensemble exceptional de $2m$-capacité nulle (capacité relative au noyau K_{2m}); g est une fonction de carré sommable. En vue de l'étude locale des fonctions de $P_0{}^m(S)$ importante dans divers problèmes, on étudie l'allure locale du potentiel précédent et d'un cas particulier qui est le potentiel $\int K_{2m}(x-y)d\mu(y)$ de la mesure positive μ lorsque sa K_{2m}-énergie est finie. On introduit une suite de mesures $\varphi_x{}^k\geq 0$ de total 1 et de support contenu dans une boule $D_x{}^{\rho_k}$ de centre x et rayon $\rho_k\to 0$ ($k\to\infty$). On cherche à ce que la moyenne pondérée (avec cette mesure) des potentiels considérés tende vers la valeur en x de ces potentiels, comme c'est le cas des moyennes sphériques des potentiels classiques. On donne des critères sur $\varphi_x{}^k$ et l'exemple du K_{2m}-potentiel de la distribution d'équilibre de la masse 1 sur un compact de $D_x{}^{\rho_k}$. On étend d'autre part la notion d'ensemble effilé en remplaçant le potentiel ordinaire par le K_{2m}-potentiel et on adapte les applications. Extension du critère du type de Wiener. *M. Brelot* (Paris).

Ivanov, V. N. The product integral and an almost periodic matrix. Dokl. Akad. Nauk SSSR (N.S.) **109** (1956), 902–905. (Russian)

The author considers the equation $\partial X/\partial t=P(t)X$, where X and P are square matrices of order n and the elements of P are uniformly almost periodic functions. The matrix $P(t)$ has a Fourier series $A+\sum A_k e^{i\delta_k t}$, where A is the constant term. A linear transformation has been carried out such that the matrix A has all its elements on one side of the diagonal equal to zero. The diagonal terms $\lambda_1, \cdots, \lambda_n$ form the matrix spectrum of P and the numbers δ_k form the exponent spectrum of P. The solution of the equation is given as a product integral

$$X=X(0)e^{At}\prod_0^t(E+e^{-At}(P(t)-A)e^{At}dt.$$

The product integral is a matrix with uniformly almost periodic elements if the set of all numbers

$$\sum_{r=1}^{s}\delta_{k_r}+\text{Im}(\lambda_r-\lambda_s),$$

where λ_r and λ_s have identical real parts and sums with $r=s$ are included, does not contain numbers in the vicinity of zero. If, further, A is a purely imaginary diagonal matrix, the solution X will have uniformly almost periodic elements. *H. Tornehave* (Virum).

Horváth, Juan. Approximation and quasi-analytic functions. Univ. Madrid. Publ. Sec. Mat. Fac. Ci. I. no. **1** (1956), 93 pp. (Spanish)

This work contains the text of the conferences in 1955 at the University of Madrid concerning results obtained on the problem of approximation of S. Bernstein. The basic ideas and generalizations are given, and applications and connections with other problems of analysis, quasi-analytic functions, and moment problems are considered. *E. Frank* (Chicago, Ill.).

See also: Manin, p. 380; Arima, p. 386; Suetin, p. 387; Bremermann, p. 387; Imšeneckaya, p. 388; Iskra, p. 390; Brunk, p. 391; Halanay, p. 395; Foguel, p. 399; Blagoveščenskii and Fil'čakov, p. 399; Park, p. 405; Kliot-Dašinskiĭ, p. 420; Hill, p. 431; Chatterjee, p. 431; Durand, p. 441.

Special Functions

Schwartz, J. Riemann's method in the theory of special functions. Bull. Amer. Math. Soc. **62** (1956), 531–540.

It was Riemann's idea to characterize a differential equation with rational coefficients in terms of its singularities and of parameters describing the behavior of its solutions at these singularities. Given an ordinary linear homogeneous nth order differential equation, the author introduces a Riemann symbol, standing for the differential equation or any one of its solutions, in order to discuss transformation, recursion, differentiation, multiplication, and other formulas for the solutions. Beyond the introduction of this symbol only the briefest outline of a method is given, the discussion consisting almost wholly of examples. *N. D. Kazarinoff.*

Tsuji, Masatsugu. Hopf's ergodic theorem on Fuchsian groups. J. Math. Soc. Japan **7** (1955), 276–289.

A Fuchsian group G in $|z|<1$ sets up a transformation group of the torus formed by all points $(e^{i\theta}, e^{i\phi})$. The author gives a new proof of E. Hopf's theorem which states that this group is metrically transitive whenever the fundamental domain D_0 of G has finite hyperbolic area. This theorem is refined to yield an ergodic property of G: For almost all θ, any line L ending at $e^{i\theta}$, and any measurable subset M of the fundamental domain, the relative hyperbolic length of L intersected with the equivalents of M tends to the hyperbolic measure of M divided by the measure of D_0. *L. Ahlfors.*

Lapin, A. I. On modular functions of degree two. Izv. Akad. Nauk SSSR. Ser. Mat. **20** (1956), 325–336. (Russian)

Extending Sel's workieg [Math. Ann. **116** (1939), 617–657] the author shows that the field of modular

functions of degree two is isomorphic to the field of rational functions of three independent variables.

A. J. Lohwater (Ann Arbor, Mich.).

Toscano, Letterio. Una generalizzazione dei polinomi di Laguerre. Giorn. Mat. Battaglini (5) **4**(84) (1956), 123–138.

A natural generalization of Laguerre polynomials to constant multiples of ${}_1F_q$'s of the form

$${}_1F_q[-n;(\alpha+1)/q,\cdots,(\alpha+q)/q;(x/q)^q]\ (q=0,1,\cdots)$$

is given. Recurrence and generating function relations are derived. The generating relation given is a special case of a relation of E. Rainville [Erdélyi et al., Higher transcendental functions, v. 3, McGraw-Hill, New York, 1955, p. 267; MR **16**, 586].

N. D. Kazarinoff.

See also: Pyateckiĭ-Šapiro, p. 378; Šidlovskiĭ, p. 382; Parodi, p. 385; Thorne, p. 393; Kostandyan, p. 432; Chakravorty, p. 434; Shibacka, p. 435; Nikitin, p. 439; Racer,Ivanova, p. 448.

Sequences, Series, Summability

Berman, D. L. On the problem of moments for a finite interval. Dokl. Akad. Nauk SSSR (N.S.) **109** (1956). 895–898. (Russian)

A sequence (1) $\{\mu_k\}$ ($k=0, 1, \cdots$) is a sequence of moments of an interval $[a, b]$, if there exists a function of bounded variation $g(x)$ such that $\int_a^b x^k dg(x)=\mu_k$. The author presents a few theorems on the conditions necessary and sufficient for (1) to be a sequence of moments, which are, in some way, more general than those given by Hausdorff [Math. Z. **16** (1923), 220–248]; also some applications are shown.

S. Kulik (Columbia, S.C.).

Epstein, M. Une généralisation des séries de Lambert. Bul. Inst. Politehn. Iaşi (N.S.) **2** (1956), 13–19. (Romanian. Russian and French summaries)

Dans cette note l'auteur donne plusieurs propositions élémentaires sur les séries $\sum_{p=1}^{\infty} a_p((1/p)\sum_{j=1}^{p} f(e_p^{j-1}z))$, où (1) f est une fonction holomorphe dans

$$\{z|\ z\neq 1,\ |z|<R,\ R>1\}$$

ayant $z=1$ pour pôle simple et s'annulant pour $z=0$; (2) e_p est une racine primitive de l'équation $z^p=1$. Pour $f(z)=z/(z-1)$ on obtient les séries de Lambert [K. Knopp, J. Reine Angew. Math. **142** (1913), 283–315].

C. T. Ionescu Tulcea (Bucarest).

Mamedov, Ya. D. Positive solutions of Uryson's integral equations. Akad. Nauk Azerbaĭdžan. SSR. Dokl. **12** (1956), 311–317. (Russian. Azerbaijani summary)

Uryson's integral equation in which the kernel depends on a parameter is discussed: $\varphi(x)=\int_a^b k(x,s;\varphi(s),\lambda)ds+f(x)$. The assumptions on $k(x, s; y, z)$ and f are: (a) k is defined for $a\leq x$, $s\leq b$, $0<y$, z; (b) k_y, k_z, $k_{yz}>0$, and these derivatives are continuous in y and z, uniformly with respect to x and s; (c) $k(x,s;0,z)=k(x,s;y,0)=0$; (d) k_{yz} is monotone decreasing in y and z and

$$k_{yz}(x,s;y_1,z_1)-k_{yz}(x,s;y_2,z_2)$$

has a positive minimum relative to x, s if $y_1<y_2$, $z_1<z_2$; (e) $\lim_{y\to\infty,z\to\infty} k_{yz}(x,s;y,z)=Q(x,s)$ uniformly, where $Q(x,s)\equiv 0$ or $\min Q(x,s)>0$; (f) $0\leq f(x)<N=\text{const}$.

If $P(x,s)=k_{yz}(x,s;0,0)$, and if α is the smallest pole of the resolvent of P, $\lambda_{y,z}$ of k_{yz} and β of Q, then $0<\alpha<\lambda_{y,z}<\beta$. Solution of the basic equation can be found by iteration according to the following scheme if $f\equiv 0$:

$$\varphi_0(x)=\int_a^b k_{yz}(x,s;y,z)\varphi_0(s)ds,$$

$$\varphi_n(x)=\int_a^b k(x,s;\varphi_{n-1}(s),\lambda)ds,\ \varphi_n\to\varphi.$$

The properties of $\varphi(x)$ as a function of λ are: $\varphi(x,\lambda)$ is monotone increasing: $\varphi(x,\lambda)\to 0$ as $\lambda\to 0$, $\varphi(x,\lambda)\to\infty$ as $\lambda\to\beta$, $\varphi(x,\lambda)$ is a differentiable function of of λ. Analogous procedures and results are obtained if f is nontrivial.

B. Gelbaum (Minneapolis, Minn.).

Iskra, K. K. Theorems on series. Grodnenskiĭ Gos. Ped. Inst. Uč. Zap. **1** (1955), 41–50. (Russian)

The (complex) sequences α and β (which are, of course, functions on the natural numbers J^+) are said to be asymptotically equivalent [in the narrow sense] if α/β and β/α are functions bounded [of bounded variation] on J^+. The ordinary comparison tests for convergence and divergence of series say essentially that if α and β are asymptotically equivalent and positive sequences, then $\sum_1^\infty \alpha_n$ and $\sum_1^\infty \beta_n$ converge or diverge together. The author here generalizes these tests by showing that if α and β are asymptotically equivalent in the narrow sense, then the same conclusion holds. This result follows from the obvious fact that if $\lambda\in \mathrm{BV}(J^+)$, then $\sum_1^\infty \lambda_n a_n$ is convergent whenever $\sum_1^\infty a_n$ is convergent. He also deduces the following analogue of the so-called MacLaurin integral test: If $\alpha\in\mathrm{BV}(J^+)$, $T\alpha/\alpha\in\mathrm{BV}(J^+)((T\alpha)_n=\alpha_{n+1})$, $\lim_{n\to\infty}\alpha_{n+1}/\alpha_n\neq -1$, $\min(\alpha_n,\alpha_{n+1})\leq\phi(x)\leq\max(\alpha_n,\alpha_{n+1})$ for $n\leq x\leq n+1$ and $n\in J^+$, and $\alpha_n=\phi(n)$ for $n\in J^+$, then $\sum_1^\infty\alpha_n$ and $\int_1^\infty\phi(x)dx$ converge or diverge together; if α is non-negative, the second and third hypotheses can be deleted.

A. E. Livingston (Seattle, Wash.).

Hlawka, Edmund. Folgen auf kompakten Räumen. Abh. Math. Sem. Univ. Hamburg **20** (1956), 223–241.

The author generalizes ideas and theorems centering around the definition by which a real sequence $x_1, x_2, \cdots$ is called uniformly distributed mod 1 if

$$\lim_{n\to\infty}\lambda_n(f)=\lim_{n\to\infty}\frac{f(x_1)+\cdots+f(x_n)}{n}=\int_0^1 f(x)dx=\mu(f)$$

whenever f is a continuous function having period 1. Instead of requiring $x_1, x_2, \cdots$ to be real numbers reduced mod 1, he requires them to be elements of a compact space X having a countable basis. Instead of considering only arithmetic means, he introduces a real matrix $A=(a_{nk})$ for which $\sum_{k=1}^\infty |a_{nk}|\leq\|A\|<\infty$ and $\sum_{k=1}^\infty a_{nk}\to 1$ as $n\to\infty$. Let $M(X)$ denote the weakly topologized space of all linear continuous functionals (or Radon measures) $\mu(f)$ defined over the vector space $C(X)$ of functions $f(x)$ which are real and continuous over X. Attention is then formed upon the linear functionals $\sum_{k=1}^\infty a_{nk}f_k(x)$ in $M(X)$ and their weak limit points. If μ is a functional in $M(X)$ and $n(1), n(2), \cdots$ is an increasing sequence of integers such that

$$\lim_{p\to\infty}\sum_{k=1}^{\infty}a_{n(p),k}f(x_k)=\mu(f)=\int_X f\,d\mu$$

for each f in $C(X)$, then μ is called an A-distribution (A-Mass) of the sequence $x_1, x_2, \cdots$. The paper contains 20 theorems involving existence, uniqueness, and other properties of these A-distributions. Several of these

involve methods and results closely related to work of J. D. Hill [Ann. of Math. (2) **46** (1945), 556–562; Pacific J. Math. **1** (1951), 399–409; **4** (1954), 227–242; MR **7**, 153; **13**, 340; **15**, 950] on the Borel property of transformations of sequences of zeros and ones. *R. P. Agnew.*

Sikkema, P. C. On sum-equations. I, II. Nederl. Akad. Wetensch. Proc. Ser. A. **59**=Indag. Math. **18** (1956), 411–421, 422–425.

Perron [Math. Ann. **84** (1921), 1–15] has related the solution of the infinite system of sum equations

$$\sum_{m=0}^{\infty}(a_m+b_{nm})x_{n+m}=c_n\ (n=0, 1, \cdots)$$

to the zeros of the function $F(z)=\sum_{m=0}^{\infty}a_mz^m$. Paasche [Über das Verhalten der Integrale homogener und inhomogener Summengleichungen im Unendlichen, Oldenbourg, München-Düsseldorf, 1954; MR **16**, 260] has established relations between the number of linearly independent solutions, the value of $\limsup_m |c_m|^{1/m}$ and the absolute values of these zeros. The first of the papers under review is concerned with the homogeneous case where $c_m=0$ and $a_1=a_2=\cdots=a_h=0$, i.e. $F(z)$ has h zero roots. It is shown that under certain conditions on the b_{nm} there exists for each of the h solutions of the homogeneous equations a positive number ρ, such that

$$\limsup_m |(m!)^{\rho}x_m|^{1/m}$$

is finite and equal to the absolute value of a zero of a certain polynomial. The second paper considers first the non-homogeneous system in which $\limsup_m |c_m|^{1/m}=0$ and relates solutions of the form $x_m=\mu_m(m!)^{-\sigma}$ to constants of the form $c_n=d_n(n!)^{-\sigma}$. It then considers the distribution of the linearly independent solutions when $a_1=a_2=\cdots=a_h=0$, and $c_n=d_n(n!)^{-\sigma}$.

T. H. Hildebrandt (Ann Arbor, Mich.).

See also: Horváth, p. 389; Smirnov, p. 398; Halilov, p. 400; Schlögl, p. 401; Kalitzin, p. 401; Wehrli, p. 428; Arakelyan, p. 434; Toraldo di Francia, p. 442; Racer-Ivanova, p. 448.

Approximations, Orthogonal Functions

Ghizzetti, Aldo. Sulle formule di quadratura. Rend. Sem. Mat. Fis. Milano **26** (1954–1955), 1–16=Consiglio Naz. Ricerche. Pubbl. Ist. Appl. Calcolo no. **434** (1956), 16 pp.

A general method based on adjoint differential forms which J. Radon [Monatsh. Math. Phys. **42** (1935), 389–396] devised for representing the remainder of interpolation and quadrature formulas is here propsed to be used for the purpose of constructing quadrature formulas. Given (i) m abscissae x_i with $a\leq x_1<x_2<\cdots<x_m\leq b$, (ii) a differential form $E_n(f)$ of order n having an adjoint $E_n^*(\varphi)$, (iii) any $m-1$ functions $\varphi_1, \varphi_2, \cdots, \varphi_{m-1}$ satisfying (almost eberywhere) $E_n^*(\varphi)$, the method leads to a formula of the type

$$\int_a^b fg\,dx=\sum_{h=0}^{n-1}\sum_{i=1}^{m}A_{hi}f^{(h)}(x_i)+R.$$

The remainder R and the coefficients A_{hi} are expressed in terms of $E_n(f)$, the adjoint forms E_n^* and the φ_i; we have $R=0$ for all f with $E_n(f)=0$. Given an integer p with $1\leq p\leq n-1$ the author studies the conditions under which a Gauss type formula (with $A_{hi}=0$ for $n-p\leq h\leq n-1$, $1\leq i\leq m$) can be obtained by a proper choice of the φ_i. He shows that this is possible in $\infty^{m(n-p)-n}$ ways, if the homogeneous differential problem $E_n(y)=0$, $y^{(h)}(x_i)=0$ ($h=0, 1, \cdots, n-p-1$; $i=1, 2, \cdots, m$) has no eigensolution. If there are q linearly independent eigensolutions Y_l, Gauss type formulas can be obtained only if $\int_a^b Y_lg\,dx=0$ ($l=1, 2, \cdots, q$), and then in $\infty^{m(n-p)-n+q}$ possible ways. Similarly, the problem of constructing Chebyshev type formulas (with $A_{hi}=0$ for $n-p\leq h\leq n-1$, $1\leq i\leq m$ and $A_{h1}=A_{h2}=\cdots=A_{hm}$ for $0\leq h<n-p$) is discussed. The method is compared with a general quadrature procedure due to Picone [Ann. Scuola Norm. Sup. Pisa (3) **5** (1951), 193–244; MR **14**, 144]. Possible extensions to multiple integrals are also indicated. *W. Gautschi.*

Brunk, H. D. On an inequality for convex functions. Proc. Amer. Math. Soc. **7** (1956), 817–824.

Let $X(t)$ be non-decreasing and continuous from the left (right) on $[a, b]$. Let $G(t)$ be of bounded variation and continuous from the right (left) on $[a, b]$, and have $G(a)=0$, $G(b)=1$. The non-decreasing approximant G^* to G is defined as follows. $G^*(t_0)=\sup_u \inf_v M\{u, v\}$, where $\{u, v\}$ denotes an arbitrary subinterval of $[a, b]$, $M\{u, v\}$ is the mean value $M\{u, v\}=\int_{\{u,v\}}G(t)dX(t)/\int_{\{u,v\}}dX(t)$, and the extrema are taken over those intervals $\{u, v\}$ that contain t_0 and have $\int_{\{u,v\}}dX(t)>0$. Slightly modified definitions of G^* are given at a and b. It is proved that $\int_{[a,b]}f[X(t)]dG(t)\geq f[\int_{[a,b]}X(t)dG(t)]$ for every continuous convex function f if and only if G^* is a distribution function on $[a, b]$, that is $G^*(a)=0$ and $G^*(b)=1$.

F. F. Bonsall (Newcastle-upon-Tyne).

Warmus, M. Calculus of approximations. Bull. Acad. Polon. Sci. Cl. III. **4** (1956), 253–259.

A theory is developed for an arithmetic and calculus of approximate numbers. The author shows that a simple and natural theory based on defining the approximate number $[a, A]$ as the closed interval $a\leq x\leq A$ leads to a system in which the distribution law fails and in which subtraction is not the inverse of addition. If however we define the approximate number $[a, A]$ to be the closed interval $\min(a, A)\leq x\leq \max(a, A)$ a more symmetrical system results in which it is possible to define the rational operations in a satisfactory way. If we write

$$[a, A]=\tfrac{1}{2}(A+a)+j\tfrac{1}{2}(A-a),$$

where j is an imaginary unit for which $j^2=1$ the arithmetic and calculus of approximate numbers is equivalent to that of the so-called hyperbolic complex numbers. A monograph on the elaboration of this theory is promised. *D. H. Lehmer* (Berkeley, Calif.).

Zuhovic'kiĭ, S. I. Some generalizations of the concept of Čebyšev alternance. Dopovidi Akad. Nauk Ukrain. RSR **1955**, 419–424. (Ukrainian. Russian summary)

Let $f(t)$ be continuous in a compact set T of real r-space, and the linear family $F_\lambda=\sum_1^n \lambda_\nu f_\nu(t)$ be continuous and unisolvent [Motzkin, Bull. Amer. Math. Soc. **55** (1949), 789–793; MR **11**, 101] there. Then λ minimizes

$$m(\lambda)=\max_t|F_\lambda-f|$$

if, for some $t_0, \cdots, t_n$ in T, $F_\lambda(t_\mu)-f(t_\mu)=\varepsilon_\mu m(\lambda)$, with $\varepsilon_\mu=\pm1$ and all equal, or all opposite, to the signs of the n-order minors of the matrix $(f_\nu(t_\mu))$ [Zuhovic'kiĭ, Mat. Sb. N.S. **33(75)** (1953), 327–342; MR **15**, 354; for special f_ν and finite T already in Kirchberger, Math. Ann. **57**

(1903), 509–540]. For $r=1$ this gives the well-known alternance property $\varepsilon_{\mu+1}=-\varepsilon_\mu$ for $t_0<\cdots<t_n$. In n-space λ is the nearest point, in the suitably weighted minimax sense, to the hyperplanes $F_\lambda=f$ (best solution of an incompatible infinite system of linear equations). For unprescribed positive weights and hyperplanes forming a simplex the λ fill the simplex; the author derives also a determinantal condition for the dual problem of a hyperplane nearest to the vertices of a simplex. {Reviewer's remarks: The latter hyperplanes form a nonconnected set; the points λ, for more than $n+1$ hyperplanes or for $n+1$ hyperplanes parallel to a line, in general form a non-convex set.} *T. S. Motzkin.*

Grebenyuk, D. G. Construction of formulas of approximate evaluation of triple integrals over an elliptic cone. Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. **16** (1955), 66–75. (Russian)

En partant de ses résultats relatifs à la représentation approchée des fonctions $f(x, y, z)$ par une classe des polynômes qui s'écartent le moins possible de zéro [mêmes Trudy **8** (1951), 45–65], l'auteur donne la formule approchée du type $\int f(x, y, z)d\sigma=\sum_1^q A_k f(x_k, y_k, z_k)+R\mu$, où l'intégrale triple est étendue à cône elliptique

$$x^2/a^2+y^2/b^2-z^2c^2=0,$$

coupé par le plan $z=h$. *M. Tomić* (Beograd).

Berman, D. L. The speed of convergence of Bernstein and Hermite-Fejer interpolation processes, constructed for certain classes of nodes. Dokl. Akad. Nauk SSSR (N.S.) **109** (1956), 1087–1090. (Russian)

The author presents a few theorems on the accuracy of the approximation of a continuous function by polynomial interpolation. The methods of interpolation considered are those of S. N. Bernstein and Hermite-Fejer. This author's method is based on the selection of nodes characteristic for the particular process of interpolation. *S. Kulik* (Columbia, S.C.).

Dzyadyk, V. K. Constructive characterization of functions satisfying the condition Lip $\alpha(0<\alpha<1)$ **on a finite segment of the real axis.** Izv. Akad. Nauk SSSR. Ser. Mat. **20** (1956), 623–642. (Russian)

While in the case of periodic functions the problem of the connection between their differential properties and the degree of their approximability by trigonometric polynomials has been essentially solved [see, e.g., D. Jackson, The theory of approximation, Amer. Math. Soc. Colloq. Publ., v. 11, New York, 1930; also Zygmund, Duke Math. J. **12** (1945), 47–76; MR **7**, 60], the results for the corresponding problem for functions on $[-1, 1]$ and their approximation by ordinary (power) polynomials do not have the same degree of finality. The present paper solves some of the outstanding problems here. 1) A necessary and sufficient condition for a function $f(x)$, $-1\leq x\leq 1$, to have a derivative $f^{(r)}(x)$ $(r=0, 1, \cdots)$ which belongs to Lip α $(0<\alpha<1)$ is that for each $n=1, 2, \cdots$ there exists a polynomial $P_n(x)$ of degree n such that

$$(*)\qquad |f(x)-P_n(x)|\leq C_n^{-r-\alpha}\{(1-x^2)^{\frac{1}{2}(r+\alpha)}+n^{-r-\alpha}\},$$

where C is a constant independent of n and x. 2) If (*) holds for $\alpha=1$, then $f^{(r)}(x)$ exists and satisfies the condition $f^{(r)}(x+h)+f^{(r)}(x-h)-2f^{(r)}=O(h)$ (the problem of the converse is still open). The proofs are based on the following lemma: 3) If a polynomial $P_n(x)$ of degree n satisfies $P_n(x)\leq\{(1-x^2)^{1/2}+n^{-1}\}^\rho$ on $(-1, 1)$ where ρ is a real number, then

$$|P_n'(x)|\leq Cn\{(1-x^2)^{1/2}+n^{-1}\}^{\rho-1},$$

where $C=C(\rho)$. *A. Zygmund* (Chicago, Ill.).

Ismaïlov, A. Ya. Estimation of derivatives of polynomials in several variables. Akad. Nauk Azerbaĭdžan. SSR. Dokl. **12** (1956), 239–243. (Russian. Azerbaijani summary)

The inequalities of S. Bernstein, A. A. Markov, S. M. Nikolsky and others for the derivatives of trigonometric and power polynomials of a single variable are extended to the case of several variables. The generalizations are straightforward. Proofs are postponed to another paper. *A. Zygmund* (Chicago, Ill.).

Il'in, V. A. Absolute and uniform convergence of expansions in eigenfunctions throughout a closed domain. Dokl. Akad. Nauk SSSR (N.S.) **109** (1956), 690–693. (Russian)

Let g be a two-dimensional region, C a contour lying strictly inside g. The author considers the expansion in eigen-functions of $(\Delta+\lambda)u=0$ over g of a class of functions, expressible as source-distributions over g and C; a particular case, it is claimed, is that of a function continuous in g, satisfying the boundary condition, with piecewise continuous first derivative, and second derivative of integrable square. Whereas in his previous paper [same Dokl. (N.S.) **105** (1955), 210–213; MR **17**, 744] the absolute and uniform convergence was for any interior sub-region of g, it is now proved for the whole closed region. On the other hand, he now restricts himself to the case of the first boundary condition, and now requires C and the boundary of g to be of "Lyapunov type". The special case in which g is a rectangle and the function expanded a constant is used to show that the result may fail if the boundary condition is not imposed on the function. *F. V. Atkinson* (Canberra).

Erdélyi, A. Asymptotic expansions of Fourier integrals involving logarithmic singularities. J. Soc. Indust. Appl. Math. **4** (1956), 38–47.

The author establishes results concerning the asymptotic behavior of $\int_\alpha^\beta \phi(t)e^{ixt}dt$ and $\int_\alpha^\beta g(t)e^{ixh(t)}dt$ for large real x. The method (of repeated integration by parts) and the results are analogous to those of his earlier paper [same J. **3** (1955), 17–27; MR **17**, 29] except that $\phi(t)$ and $g(t)$ are here allowed to have logarithmic singularities. For example, in one of the three theorems,

$$\phi(t)=\phi_1(t)(t-\alpha)^{\lambda-1}\log(t-\alpha),$$

where $0<\lambda<1$ and $\phi_1(t)$ is N times continuously differentiable for $\alpha\leq t<\beta$. *T. E. Hull* (Vancouver, B.C.).

See also: Horváth, p. 389; Berman, p. 390; Džrbašyan, p. 393; Grebenyuk, p. 394; Wintner, p. 396; Schlögl, p. 401; Mikeladze, p. 419; Kunz, p. 419.

Trigonometric Series and Integrals

Sargsyan, I. S. Theorems concerning summation of derivatives of expansions into the standard and the generalized Fourier integral. Akad. Nauk Armyan. SSR. Dokl. **23** (1956), 3–10. (Russian. Armenian summary)

The main result of this note is the author's proof of the

Fejer-Lebesgue theorem on the summation of the derivatives of the Fourier series by the Riesz method. A theorem on the summation of derivatives of the Fourier integral of a function is presented with the reference to the above proof. *S. Kulik* (Columbia, S.C.).

Kalitzin, Nikola St. Fourierreihen und Randwertaufgaben. Univ. d'Etat Varna Fac. Tech. Constructions. Annuaire 4 (1948–1949), 1–45. (Bulgarian. German summary)

An exposition is given for a Fourier series solution of certain linear partial differential equations with boundary conditions. *P. Civin* (Eugene, Ore.).

Dživarševili, A. G. On the summation of conjugate series and series of Fourier-Denjoy. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze 22 (1956), 203–225. (Russian)

Applications of general linear methods of summation to Fourier-Denjoy series and their conjugates. The main tools are the facts that a Fourier-Denjoy series is a termwise differentiated Fourier series of a function differentiable almost everywhere, and that the formula of integration by parts holds for Denjoy integrals.

A. Zygmund (Chicago, Ill.).

Timan, M. F. On absolute summability of Fourier series in two variables. Soobšč. Akad. Nauk Gruzin. SSR 17 (1956), 481–488. (Russian)

A study of necessary and sufficient conditions for the Fourier series of a function of two variables to be absolutely summable $|C, \alpha, \beta|$. The results follow the pattern established for the case of functions of a single variable by L. Bosanquet [Proc. London Math. Soc. 41 (1936), 489–512]. *A. Zygmund* (Chicago, Ill.).

Džrbašyan, M. M. On the theory of series of Fourier in terms of rational functions. Akad. Nauk Armyan. SSR. Izv. Fiz.-Mat. Estest. Tehn. Nauki 9 (1956), no. 7, 3–28. (Russian. Armenian summary)

The author investigates the problem of functions defined on the unit circle into the Fourier series of a set of orthogonal rational functions, with a given set of poles. He derives an expression of Dirichlet's kernel, which is similar to that of trigonometric functions. The problem of convergence is discussed, and, in particular, it is shown that, if the poles of the orthogonal set of functions do not cluster at the unit circle, the character of convergence is the same as in the case of trigonometric Fourier Series.

S. Kulik (Columbia, S.C.).

See also: Vinogradov, p. 381; Nikolaeva, p. 385; Ismaĭlov, p. 392; Erdélyi, p. 392; Blagoveščenskiĭ, and Fil'čakov, p. 399; Kostandyan, p. 432; Nazarov, p. 433; Koiter, p. 435.

Integral Transforms

Bolinder, E. Folke. The relationship of physical applications of Fourier transforms in various fields of wave theory and circuitry. Acta Polytech. no. 189 (1956), i+22 pp.

A procedure is presented for connecting some known physical applications of Fourier transform pairs in different branches of the theory of waves and circuitry. After an investigation of the cases of diffraction, reflection, and coupling of waves, deflection of particles (which includes the cathode-ray-tube case and the so-called gap effect) and the closely related scanning problem are examined. Finally, the extension to random functions is discussed briefly. *Author's summary.*

See also: Donegan and Huss, p. 394; Beattie and Davies, p. 400; Bogolyubov and Parasyuk, p. 404; Park, p. 405; Choudhury, p. 434; Koiter, p. 435.

Ordinary Differential Equations

Wintner, Aurel. On certain absolute constants concerning analytic differential equations. Amer. J. Math. 78 (1956), 542–554.

Let $f(z, w)$ be a regular function of two complex variables satisfying $|f(z, 0)| \leqq N$, $|f_w(z, w)| < L$ for $|z| < a$, $|w| < b$. Then the initial value problem $dw/dz = f(z, w)$, $w(0) = 0$ has a solution $w = w(z)$ on the circle

$$|z| < \min (a, L^{-1} \log (1 + bL/N))$$

[cf. Painlevé, Bull. Soc. Math. France 27 (1899), 149–152]. Wintner shows that this result cannot be improved.

Let $f(w) = \sum c_n w^n$ be regular and satisfy $|f(w)| < 1$ on $|w| < 1$. Let $f^*(w) = \sum |c_n| w^n$. The author points out the existence of absolute constants γ (independent of f) such that $dw/dz = f^*(w)$, $w(0) = 0$ has a solution $w = w(z)$ on $|z| < \gamma$. He does not determine the "best" value of γ (which is $\gamma = \inf \int_0^1 dw/f^*(w)$) but obtains lower estimates (first $\gamma \geqq \frac{1}{3}$, second $\gamma \geqq \frac{1}{4}\pi$) from two different majorants for $|f^*|$. (Incidentally, a combination of his majorants leads to a better lower bound for γ, so that $\frac{1}{4}\pi$ is not the "best".) {His theorem (I) is incorrectly stated; equation (22) should be replaced by $\gamma = \frac{1}{4}\pi$. Wintner treats a similar question where $f(w)$ is replaced by $f(z, w)$; the author's theorem (II) is also incorrectly stated.}

P. Hartman (Baltimore, Md.).

★ **Thorne, R. C. The asymptotic solution of linear second order differential equations in a domain containing a turning point and regular singularity.** Department of Mathematics, California Institute of Technology, Technical Report 12, Office of Naval Research NR 043-121, 1956. pp. i+1–21 (bound with Report 13).

Let u be a large, positive parameter and z be a complex variable confined to an open simply connected domain D containing $z=0$ and a point $z=z_0$. Let $p(z)$ and $q(z)$ be regular in D with the property that p and q are real when z is real. Under these assumptions, asymptotic expansions of solutions of the differential equation

$$(1) \qquad d^2w/dz^2 = [u^2(z_0 - z)z^{-2}p(z) + z^{-2}q(z)]w$$

are derived which are uniformly valid with respect to z in domains including $z=0$ and $z = z_0 \pm i$, 0 when $u \to \infty$. The expansions involve Bessel functions of large order whose argument may be either large or small. The investigation rests heavily upon work of Olver [Philos. Trans. Roy. Soc. London. Ser. A. 249 (1956), 65–97; MR 18, 38]. For this reason many proofs are suppressed. The fundamental comparison equation to which (1) is related is $d^2y/dt^2 = [u^2(1 + \alpha^2 t^{-2}]) - (2t)^{-2}]y$; $\alpha = m/u$ and is fixed. The points $t = \pm i\alpha$ correspond to $z = z_0 \pm i \cdot 0$, respectively. The solutions of this equation have the form $t^{\frac{1}{2}}\mathscr{C}_m(ut)$, where $\mathscr{C}_m$ is a cylinder function of order m.

N. D. Kazarinoff (Ann Arbor, Mich.).

★ **Markus, L. Asymptotically autonomous differential systems.** Contributions to the theory of nonlinear oscillations, vol. 3, pp. 17–29. Annals of Mathematics Studies, no. 36. Princeton University Press, Princeton, N. J., 1956. \$4.00.

The author studies the topological character of the limiting sets of solutions of (1) $\dot{x}=f(x, t)$ (x a real n-dimensional vector, t a real variable) by relating them to those of solutions of a limiting equation (2) $\dot{x}=f(x)$, where $\lim_{t\to\infty} f(x, t)=f(x)$ uniformly in x on each compact subset of some open set in E^n. Some of the results are applied to the generalized van der Pol equation

$$\ddot{x}+\mu(t)f(x)\dot{x}+g(x)=0,$$

where among other conditions $\lim_{t\to\infty}\mu(t)=\mu_\infty>0$.

C. E. Langenhop (Ames, Ia.).

Tatarkiewicz, Krzysztof. Quelques exemples de l'allure asymptotique des solutions d'équations différentielles. Ann. Univ. Mariae Curie-Skłodowska. Sect. A. **8** (1954), 105–133 (1956). (Polish and Russian summaries)

The author investigates the weakest hypotheses that imply various asymptotic properties of solutions of the equation (1) $\dot{x}=[a(t)+b(t)]x+f(t)$, where $a(t)$, $b(t)$, and $f(t)$ are continuous on $[0, +\infty)$. He also demonstrates by means of counter examples which hypotheses are essential. Suppose that for each $\varepsilon>0$ there exists an $M_\varepsilon>0$ such that (2) $\int_0^\infty |f(t)| \exp(-\varepsilon t)dt\leqq M_\varepsilon$ and that (3) $\lim_{t\to\infty} t^{-1}\int_0^t b(s)ds=0$. Then if

$$\limsup_{t\to\infty} a(t)\leqq 0,$$

all solutions $x(t, c)$ of (1) have the property that (4) for each $\varepsilon>0$, $\lim_{t\to\infty} x(t, c)\exp(-\varepsilon t)=0$; if $\liminf_{t\to\infty} a(t)>0$, there exists exactly one solution $x(t, k)$ of (1) having property (4) and all others grow faster in absolute value than $e^{\varepsilon t}$, $\varepsilon<\varepsilon_0$. This is the principal theorem of the paper. The second mean value theorem of integral calculus is the main tool used in the proof. The equation $\dot{x}=-[\sin\ln t+\cos\ln t]x+1$ satisfies all the hypotheses of the theorem except (3), and none of its solutions have property (4). Suppose $\int_0^t b(s)ds$ and $\int_0^t f(s)ds$ are bounded on $[0, \infty)$. Then if $a(t)\leqq 0$, all solutions of (1) are bounded; if $a(t)\geqq 0$ and $\int_0^\infty a(t)dt=+\infty$, then exactly one solution of (1) is bounded, and it even converges to zero as $t\to\infty$. If $|f(t)|\leqq M$ and $a(t)+b(t)\leqq -\lambda<0$, all solutions of (1) are bounded; if $|f(t)|\leqq M$, and $a(t)+b(t)\geqq\lambda>0$, exactly one solution of (1) is bounded. The proof of this theorem is based upon the retraction theorem of Ważewski [Ann. Soc. Polon. Math. **20** (1947), 279–313; MR **10**, 122]. Generalizations of the main theorem to the case where condition (2) is replaced by $\int_0^\infty |f(t)|\exp(-\varphi(\varepsilon, t))dt\leqq M_\varepsilon$, for some function φ, are considered, and several examples are given illustrating what can happen in the absence of various hypotheses. Second order equation and nonlinear ones of the form $\dot{x}=a(t)x+d(x, t)+f(t)$, $d(0, t)=0$, are touched upon. References to the literature are made throughout, especially in connection with the numerous examples.

N. D. Kazarinoff (Ann Arbor, Mich.).

Chin, Yuan-Shun. Limit cycles with even multiplicities. Acta Math. Sinica **5** (1955), 269–282. (Chinese. English summary)

Consider the differential system

$$(S_\lambda)\ \frac{dx_1}{dt}=X_1\cos\lambda-X_2\sin\gamma,\ \frac{dx_2}{dt}=X_2\cos\lambda+X_1\sin\lambda,$$

where X_1 and X_2 are real analytic functions of x_1 and x_2 (the author informed the reviewer that the original assumption that X_1 and X_2 are of class C^1 is not strong enough for the validity of Lemma 2.3), and λ is a real parameter. Let R be a ring region bounded by two simple closed curves C_1 and C_2 which (1) posesss continuously turning tangents and (2) are tangent to no integral curves of S_0; furthermore let R, C_1 and C_2 contain no singular points of S_0. The main results are: (I) If R contains a limit cycle C_0 of S_0 with even multiplicity, then there exists a η such that for $0<\lambda<\eta$, R contains two limit cycles of S_λ, or of $S_{-\lambda}$ of odd multiplicities, one stable and one unstable, and C_0 lies between them; (II) If R contains no limit cycle of S_0 of odd multiplicity and if there is a η such that for $0<\lambda<\eta$, R contains a limit cycle of S_λ or $S_{-\lambda}$ of odd multiplicity, then R contains a limit cycle of S_0 of even multiplicity. Using these results the author shows that the problem of locating the limit cycles of S_0 with even multiplicities can be reduced to that of S_λ with odd multiplicities to which the classical method of Poincaré applies.

C. T. Taam (Washington, D.C.).

Chin, Yuan-Shun. Sur les cycles limites multiples. Acta Math. Sinica **5** (1955), 243–252. (Chinese. French summary)

The author investigates the properties and location of limit cycles of the real differential system

$$(1)\qquad \frac{dx_1}{dt}=X_1(x_1, x_2),\ \frac{dx_2}{dt}=X_2(x_1, x_2),$$

where X_1 and X_2 are of the class C^1. Let C be a limit cycle of (1) and let AB be an arc which is tangent to no integral curves of (1) and which meets C at p; let also AB contain no singular points of (1); take a positive semi-orbit of (1) which crosses AB first at q_1 and then at q_2 consecutively; define a functions $\phi(S_1)=S_2$, S_i being the directed arc length of pq_i along AB; the order of the contact of the curve $\phi(S_1)=S_2$ with the line $S_1=S_2$ at $(0, 0)$ is called the multiplicity of C. With the aid of a result of S. P. Diliberto [Contributions to the theory of nonlinear oscillations, Princeton, 1950, pp. 1–38; MR **11**, 665] the author finds $\phi'(0)=\exp\int_C \operatorname{div} Xdt$, where $X=(X_1, X_2)$. It follows that a limit cycle C has multiplicity greater than 1 if and only if $\int_C \operatorname{div} Xdt=0$. As corollaries, methods of locating limit cycles with multiplicity greater than 1 are obtained. Using Green's theorem, the author proves also that a periodic orbit C is stable (unstable) on one side if $\operatorname{div} X<0$ (>0) on that side in the neighborhood of C.

C. T. Taam.

Donegan, James J.; and Huss, Carl R. Incomplete time response to a unit impulse and its application to lightly damped linear systems. NACA Tech. Note no. 3897 (1956), 17 pp.

Grebenyuk, D. G. Formulas of approximate representation of solutions of the equation $(ax+b)y''+c_1y'+c_0y=0$ as polynomials of the second degree on the interval $[-1, +1]$. Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. **16** (1955), 45–65. (Russian)

Cette note représente une application des résultats de l'auteur [Trudy Sredneaz. Gos. Univ. Ser. 5, Mat. **18** (1939)] relatifs aux polynômes qui s'écartent le moins possible de zéro, et dont les coefficients satisfont à plusieurs relations linéaires. Pour l'équation en question, avec les conditions aux limites $x=x_0$, $y=y_0$, $y'=y_0'$ $\epsilon[-1, 1]$, $-b/a\notin[-1, 1]$, on obtient la solutionap prochée sous la forme d'un tel polynôme du deuxième degré. On

discute ensuite les différents cas de cette solution approchée suivant la position et le nombre de certaines valeurs caractéristiques de ce polynôme. *M. Tomić.*

★ **Marcus, Marvin D. Repeating solutions for a degenerate system.** Contributions to the theory of nonlinear oscillations, vol. 3, pp. 261–268. Annals of Mathematics Studies, no. 36. Princeton University Press, Princeton, N. J., 1956. $4.00.

The author considers the system

$$\dot{x}_i=1+kP_i(x_1, x_2, x_3, k)\ (i=1, 2),$$
$$\dot{x}_3=k[A(x_1, x_2)x_3+Q(x_1, x_2, x_3, k)],$$

where x_i, $k \in E^1$, $x_3 \in E^2$ and A, P_i, Q are of class C^3 for all values of their arguments and of period ω_i in x_i. Under the hypotheses:

$$P_i=O(\|x_3\|+|k|),\ Q=O(\|x_3\|^2+|k|),$$
$$\sup\{\|(I+kB(s))^{-1}\|\ |s \in [0, \omega_1]\}\leqq(1+kK)^{-1},$$

where $B(s)=\int_0^{\omega_2} A(t+s, t)dt$ and $K>0$, he proves, using Schauder's fixed-point theorem, that there is for sufficiently small positive k a repeating solution which reduces to $x_1=t+s$, $x_2=t$, $x_3=0$ when $k=0$.

H. A. Antosiewicz (Washington, D.C.).

Šimanov, S. N. On the problem of finding the characteristic exponents of systems of linear differential equations with periodic coefficients. Dokl. Akad. Nauk SSSR (N.S.) **109** (1956), 1102–1105. (Russian)

Consider the n-vector system

$$(1)\qquad \dot{x}=(A+\mu F(t, \mu))x,$$

where A is a constant matrix and F has period 2π in t and is analytic in μ (real) in some interval $|\mu|<\mu^*$. Let λ_1 be a characteristic root of A. The analytic nature of the characteristic exponent of (1) corresponding to λ_1 a simple root has been discussed by Artemiev [Izv. Akad. Nauk SSSR. Ser. Mat. **8** (1944), 61–100; MR **6**, 189] and Shimanov [Prikl. Mat. Meh. **16** (1952), 129–146; MR **13**, 745]. The present note takes up the case where λ_1 is multiple or even where two λ's differ by a multiple of i. The nature of the fractional powers of λ_1 in the expansions of the characteristic exponents is fully determined.

S. Lefschetz (Mexico, D.F.).

Garnier, René. Sur les systèmes différentiels Σ, à points critiques fixes, associés au problème de Riemann pour les systèmes linéaires d'ordre ≥ 2. Rend. Circ. Mat. Palermo (2) **5** (1956), 73–92.

Let $\{t_i|i=1, \cdots, n\}\cup\{0\}\cup\{1\}$ be $n+2$ assigned regular singularities, in the complex plane, of the matrix system (1) $\dfrac{dy}{dt}=yA$, where y is an m vector and A is the sum of $n+2$ constant matrices each of the form $A_i(x-t_i)^{-1}$. The Riemann problem for (1) is that of finding A_i so that the monodromy group of the solutions of (1) is an assigned group G. This leads to the system

$$(2)\qquad \frac{dA_h}{dt_i}=(\delta_{hi}-1)A_iA_h/(t_h-t_i)$$

with δ the Kronecker symbol. The author supposes the Riemann problem solved where the elements of G induce linear transformations of the solutions with coefficients involving $\exp 2\pi i r_j$ for real r_j in the case considered. The contribution of the paper is the study of the behaviour of the matrix solutions of (2) when some of the t_i's coalesce in terms of expansions in powers of t^s, s depending linearly on $\{r_j\}$.

D. G. Bourgin (Urbana, Ill.).

Halanay, A. Almost-periodic solutions of certain nonlinear systems. Gaz. Mat. Fiz. Ser. A. **7** (1955), 396–399. (Romanian. Russian and French summaries)

The following two theorems are proved: 1. Let $u(t)$ be a bounded solution of a periodic system (with period ω); if all the solutions $u(t+k\omega)$ are stable in the sense of Lyapunov, uniformly with respect to k, then u is asymptotically almost-periodic and the system admits an almost-periodic solution. 2. If a positive semi-trajectory of a dynamical system is stable in the sense of Lagrange and uniformly stable in the sense of Lyapunov, it is asymptotically almost-periodic.

J. L. Massera.

Troickiĭ, V. A. On the problem of self-vibrations in systems of automatic regulation with two servomotors of constant velocity. Prikl. Mat. Meh. **20** (1956), 627–638. (Russian)

The basic equation is

$$(1)\qquad \dot{x}=bx+h_1f_1(\sigma_1)+h_2f_2(\sigma_2),\ \sigma_i=j_ix.$$

Here x, h_i are n-(column)vectors, the j_i are constant row-vectors, b is a constant $n\times n$ matrix and the f_i are scalar functions: the characteristics of the two servos. These are assumed of "on-off" type, so that the f_i have each just one jump and are constant otherwise. Assuming coordinates such that b is in normal form the calculations are carried out in detail in the obvious way. Same treatment for servos with constant speed. The equations are then

$$\dot{x}=bx+n_1\xi_1+n_2\xi_2,$$
$$\dot{\xi}_i=f_i(\sigma_i),\ \sigma_i=j_ix+r_{i1}\xi_1+r_{i2}\xi_2.$$

S. Lefschetz (Mexico, D.F.).

Volk, I. M. On a class of self-oscillating systems. Dokl. Akad. Nauk SSSR (N.S.) **110** (1956), 189–192. (Russian)

Let x, y be scalar variables, z an m-vector, c and μ small scalar parameters. Consider the system

$$\dot{x}=X(x, y)+\mu\{P(\mu, x, y)+cU(c, \mu, x, y, z)\},$$
$$(1)\qquad \dot{y}=Y(x, y)+\mu\{Q(\cdots)+cV(\cdots)\},$$
$$\mu^{-k}\dot{z}=Az+R(x, y)+Z(c, \mu, x, y, z),$$

where A is a constant matrix and k an arbitrary (positive or negative) integer. It is assumed that

$$(2)\qquad \dot{\xi}=X(\xi, \eta),\ \dot{\eta}=Y(\xi, \eta)$$

has a first integral independent of t. The author studies the appearance of steady state self-oscillations in (1) under these assumptions: (a) If $H(\xi, \eta)$ is the assumed integral of (2) then some curve $\gamma{:}H=h$ is closed and along it

$$(X^2+Y^2)\left(\left(\frac{\partial H}{\partial \xi}\right)^2+\left(\frac{\partial H}{\partial \eta}\right)^2\right)\neq 0;$$

(b) no characteristic root of A is zero and for $k=0$ none is a multiple of $2\pi i T^{-1}$, where

$$T=\oint_\gamma (X^2+Y^2)^{-\frac{1}{2}}ds.$$

When (a), (b) hold the system consisting of (2) together with

$$(3)\qquad \dot{\zeta}=\mu^k(A\zeta+R(\xi, \eta))$$

has a unique closed solution $\Gamma(\mu)$ continuous in μ on a certain interval around zero, projected onto γ in the plane (ξ, η). Let Γ_0 correspond to $\mu=0$. Additional assumption: (c) In a certain ring around Γ_0 the right hand sides of (1) are continuous in μ and holomorphic in the other variables for c small enough. Two other complicated conditions being fulfilled, the author states (without proof) that (1) has a limit-cycle $\Gamma(\mu, c)$ depending continuously on μ, c and $\to\Gamma_0$ as both $\to 0$. Under the same conditions the period of the corresponding solutions $\to T$.

S. Lefschetz (Mexico, D.F.).

Zubov, V. I. A qualitative study of a system of ordinary differential equations. Dokl. Akad. Nauk SSSR (N.S.) **109** (1956), 899–901. (Russian)

The object of this note is the determination of conditions for asymptotic stability in the large of the origin for a system

$$\dot{x}=f_1(x, y),\ \dot{y}=f_2(x, y),\ \dot{z}=f_3(x, y, z)$$

under the following conditions: (a) The f_i satisfy the usual continuity and unicity assumptions for all x, y, z. (b) $f_i=0$, $i=1, 2$, determines a curve s_i in the x, y plane which divides the latter into two parts, in one of which $f_i>0$, and in the other $f_i<0$. (c) The curves $s_i=0$ intersect at the origin and nowhere else, and $f(0, 0, z)=0$ only if $z=0$. (d) There is no closed solution of $x\,dy-y\,dx=0$. (e) $f_3=0$ defines in $z^2+y^2\leqq\rho^2$ a continuous function $z=\varphi(x, y)$. (f) $(\varphi-z)/f_3>0$ for $z\neq\varphi$. (g) The origin is asymptotically stable. Under these conditions the author proves several rather complicated theorems the simplest of which is: Suppose that the domain A of asymptotic stability is not the whole space. Let also $f_1=0$, $f_2=0$ define single valued functions $y_1(x)$ for $x\geqq c_1$ and $y_2(x)$ for $x\leqq c_2$ and similar functions $x_i(y)$; then the boundary of A consists of at most four cylindrical surfaces.

S. Lefschetz (Mexico, D.F.).

Zubov, V. I. Investigation of the neighborhood of the equilibrium position of a system of differential equations. Dokl. Akad. Nauk SSSR (N.S.) **110** (1956), 169–171. (Russian)

Generalizing the situation of the preceding paper the author considers the n-vector system

$$(1)\qquad \dot{x}=\sum_{m=\mu}^{\infty} X^{(m)}(x, t);$$

let the components of $X^{(m)}$ be forms of degree m in those of x, the coefficients being bounded for $t\geqq 0$, and the series convergent for x small enough and $t\geqq 0$. The author asks under what conditions the origin is asymptotically stable for given $X^{(\mu)}$ independently of the $X^{(\mu+h)}$. This is made to depend upon the asymptotic stability of the origin for

$$(2)\qquad \dot{x}=f(x),\ f(0)=0,$$

where the continuity and unicity conditions hold for all x. Here two theorems are stated: Theorem 1. Necessary and sufficient conditions for the asymptotic stability of the origin for (2) are: (a) there is an O^+-curve and no O^--curve ($O^\pm$-curve: $\to$ origin as $t\to\pm\infty$); (b) the origin has a neighborhood free from complete trajectories of (2). Theorem 2 gives sufficient conditions for the existence of $O^\pm$-curves for (1). Two complicated theorems are stated answering the original question. There are no proofs.

S. Lefschetz (Mexico, D.F.).

***Малкин, И.Г.** [Malkin, I. G.] **Некоторые задачи теории нелинейных колебаний. [Some problems of the theory of nonlinear oscillations.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1956. 492 pp. 16 rubles.

This is a book written with a great deal of care by one of the outstanding Soviet masters of the subject. It stands somewhere between the text book "for fairly advanced students" level and the level "for advanced specialists." Furthermore it addresses itself not only to the strict mathematician — to whom it has much to say, but also to the student of applied mathematics. Withal it never relaxes its mathematical vigilance. One of the distinguishing features of this book is its detailed attention to methods of calculation. For instance in dealing with periodic solutions of resonant non-autonomous systems 5 pages are devoted to the calculation of the solutions for analytic systems, 12 more for non-analytic systems including a detailed proof of the convergence of the successive approximations which are used in this case. A number of detailed technical examples are discussed throughout. Noteworthy also is an extensive treatment — we believe the first in a treatise — of almost periodic oscillations. {An objectionable feature, dictated perhaps by didacticism, is the failure to make an energetic use of vector-matrix notations with resulting complications that one may well guess. This is however a general failing in practically all the Soviet writings on differential equations. The absence of a bibliography is also to be noted but there are abundant footnotes throughout the book.}

S. Lefschetz (Mexico, D.F.).

Thomas, Johannes. Über die Existenz unendlich vieler reeller Eigenwerte bei einer gewissen Klasse Sturmscher Differentialgleichungen nebst gewissen nicht selbstadjungierten Nebenbedingungen. Math. Nachr. **14** (1955), 235–239 (1956).

In the differential equation $(f(x)y')'+(\lambda g(x)+h(x))y=0$, let $f(x)>0$, $g(x)$, $h(x)$ be real continuous functions on $-a\leqq x\leqq a$. The author considers eigenvalue problems determined by this equation and one of the boundary conditions: (i) $y(-a, \lambda)=0$ and $\int_{-a}^{a} A(x)y(x, \lambda)dx=0$, where A is real and L-integrable; (ii) same as (i) except that the condition $y(-a, \lambda)=0$ is replaced by $y(a, \lambda)=0$; (iii) a certain condition involving several points of the interval. In particular, the author shows that f and h are even, g is odd, and A is either odd or even on $-a\leqq x\leqq a$, then, corresponding to either of the boundary conditions (i) or (ii), there exists an infinity of real eigenvalues λ (with no finite cluster point). Some example are mentioned.

C. R. Putman (Lafayette, Ind.).

See also: Salié, p. 379; Ivanov, p. 389; Schwartz, p. 389; Reeb, p. 407; Laasonen, p. 418; Goodman and Lance, p. 420; Yanowitch, p. 423; Sideriades, p. 428; Wehrli, p. 428; Išlinskiĭ, p. 428; Morozova, p. 428; Aĭzerman and Gantmaher, p. 453.

Partial Differential Equations

Wintner, Aurel. On the regularity regions of the solutions of the partial differential equations of Cauchy-Kowalewsky. Amer. J. Math. **78** (1956), 525–541.

Let $f(z, w, s)$, $g(z, w, s)$ be analytic functions of three complex variables satisfying $|f|<L$, $|g|<M$ on the (z, w, s)-domain $|z|<a$, $|w|+L|z|<b$, $|z|<c$. The main result of the paper is the statement that the quasi-linear

Cauchy problem $s_z=fs_w+g$, $s(0, w)=0$ has a solution $s=s(z, w)$ on the domain $|z|<\min(a, c/M)$. $|w|+L|z|<b$ and that this domain cannot be improved in terms of absolute constants. The proof follows that of Hartman and Wintner [same J. **74** (1952), 834–864; MR **14**, 475] and depends on the method of successive approximations as initiated by Perron in the real field. Also, arguments involving normal families are used to deduce the sharp estimate for the size of the domain of existence. The last part of the paper deals with estimates for $R=R_C(r)$, where R is the largest number for which the Cauchy problem $\varphi_z=C/(1-z)(1-w)(1-\varphi)(1-\varphi_w)$, $\varphi(0, w)=0$ has an analytic solution $\varphi=\varphi(z, w)$ on $|z|<R$, $|w|<r$.

P. Hartman (Baltimore, Md.).

Bellman, Richard; and Wing, G. Milton. Hydrodynamical stability and Poincaré-Lyapunov theory. I. Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 867–870.

Consider the equation (a) $\partial\theta/\partial t+q(\partial\theta/\partial x)=1$ $(t>0;$ $0\leq x\leq 1)$, (b) $\int_0^1 R(\theta, q)dx=1$, where $\theta=\theta(x, t)$, $q=q(t)$ and $R(\theta, q)$ satisfies some smoothness properties which are too detailed to mention. The authors investigate the stability of the solution (c) $\theta_0=a_0+xq_0^{-1}$, $q=q_0$, where q_0 is a constant and a_0 is chosen so as to satisfy (b). Using a method of successive approximations, it is shown that if the roots of the characteristic equation of the linear variational equation have real part negative, then the solution (c) is asymptotically stable in the sense of Lyapunov. *J. K. Hale* (St. Paul, Minn.).

Lax, Anneli. On Cauchy's problem for partial differential equations with multiple characteristics. Comm. Pure Appl. Math. **9** (1956), 135–169.

Ce travail est consacré aux équations aux dérivées partielles linéaires $L(\partial/\partial x, \partial/\partial t)\mu=0$, d'ordre m à deux variables indépendantes. Pour ces équations, le problème de Cauchy consiste á rechercher u connaissant cette fonction et ses dérivées normales d'ordre inférieur à m sur une courbe, dite courbe initiale, qui n'est ni caractéristique, ni tangente à une caractéristique. Ce problème est dit proprement résoluble s'il existe une valeur de l'entier k telle que tout système de données de Cauchy k fois dérivables détermine une solution unique d'un côté de la courbe initiale.

Le présent mémoire recherche des critères algébriques commodes permettant d'affirmer que le problème de Cauchy est proprement résoluble.

La première partie concerne les équations à coefficients constants. Désignons par $a_{i,j}$ le coéfficient de $\partial^{i+j}u/\partial x^i dt^j$ dans L et posons

$$p_k(z)=a_{0,k}z^k+a_{1,k-1}z^{k-1}+\cdots+a_{k,0}\ (0\leq k\leq m).$$

Alors, le problème de Cauchy posé pour L est proprement résoluble si et seulement si (a) $p_m(z)$ n'a que des racines réelles et, ou bien (b) $(z-\lambda)^{r-s}$ divise $p_{m-s}(z)$ pour $s=0, 1, 2, \cdots, r-1$, r désignant le plus grand entier tel que $(z-\lambda)^r$ divise $p_m(z)$, ou bien (b') si le plus commun diviseur de $p_m(z)$, $\partial p_m(z)/\partial z$, $\cdots$ et $\partial^k p_m(z)\partial z^k$ divise $p_{m-k}(z)$, pour $k=1, 2, \cdots, m-1$.

Dans le cas de deux variables indépendantes, ce critère équivaut au critère général de Gårding [Acta Math. **85** (1951), 1–62; MR **12**, 831] mais s'avère plus facile à vérifier. Les démonstrations établissent en même temps que le domaine de dépendance d'un point P est justement la portion de la courbe initiale découpé par les deux caractéristiques les plus extérieure issues de P.

La condition nécessaire et suffisante est étendue aux équations à coefficients variables dans la seconde partie du travail. *H. G. Garnir* (Liège).

Miles, E. P., Jr.; and Williams, Ernest. A basic set of polynomial solutions for the Euler-Poisson-Darboux and Beltrami equations. Amer. Math. Monthly **63** (1956), 401–404.

The wave equation, the Laplace, Euler-Poisson-Darboux and Beltrami equations are written in the one formula

$$E_{ij}=\sum_{n=1}^{m} u_{x_n x_n}+(-1)^i[u_{tt}+jkt^{-1}u_t]=0\ (i=0, 1; j=0, 1),$$

where k is a real constant. From the basic sets of polynomials given in a former article [Proc. Amer. Math. Soc. **6** (1955), 191–194; MR, **17**, 252] for E_{10} and E_{00}, the authors deduce basic sets for E_{11} and E_{01} if $k>0$, and in the case of E_{11} they point out the association with a certain type of Cauchy problems. *M. J. De Schwarz.*

Miles, E. P., Jr.; and Williams, Ernest. The Cauchy problem for linear partial differential equations with restricted boundary conditions. Canad. J. Math. **8** (1956), 426–431.

The authors express the solutions of certain, frequently encountered types of Cauchy problems concerning linear partial differential equations of arbitrary order s and of the form

$$\Phi(D_1x_1, x_2, \cdots, x_m)u+\Psi(D, t)u=0$$

by a set of solutions of systems of ordinary differential equations. These results allow to derive basic sets of polynomials established by the authors in a previous paper [Proc. Amer. Math. Soc. **6** (1955), 191–194; MR **17**, 252] in a more natural way and are illustrated by examples concerning the wave equation and the Euler-Poisson-Darboux equation. Reference is made to a result of A. Weinstein [Proc. Symposia Appl. Math., v. 5, McGraw-Hill, New York, 1954, pp. 137–147; MR **16**, 137].

M. J. De Schwarz (Rome).

Aržanyh, I. S. On the method of characteristics for simultaneous partial differential equations of the first order. Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. **16** (1955), 23–27. (Russian)

Verfasser behandelt Systeme der Art

$$F_\rho(x_1, \cdots, x_n, p_1, \cdots, p_n)=0\ (\rho=1, 2, \cdots, r),$$

worin p_ν die ersten Ableitungen der unbekannten Funktion u bedeuten $(\nu=1, 2, \cdots, n)$. Das System wird durch die Klammerbedingungen

$$(F_\rho, F_\sigma)=0\ (\rho, \sigma=1, 2, \cdots, r)$$

ergänzt. Im Sinne des Cauchyschen Problems wird die Anfangsmannigfaltigkeit

$$X_\nu=\varphi_\nu^{(0)}(s_1, \cdots, s_{n-r}),\ u=\varphi^{(0)}(s_1, \cdots, s_{n-r})$$

vorgeschrieben. Die Lösungen der Differentialgleichungen

$$(*)\quad dx_\nu=\sum_\varrho \frac{\partial F_\varrho}{\partial p_\nu}dt_\rho,\ dp_\nu=-\sum_\varrho \frac{\partial F_\varrho}{\partial x_\nu}dt_\varrho,\ du=\sum_{\varrho,\nu}\frac{\partial F_\rho}{\partial p_\nu}p_\nu dt_\varrho$$

heiszen charakteristische Mannigfaltigkeiten. Im Sinne des Cauchyschen Problems fallen die charakteristischen Fortschreitungsrichtungen $dt_1:dt_2:\cdots:dt_\rho:\cdots:dt_r$ mit keiner der Tangentialrichtungen auf der vorgegebenen Anfangsmannigfaltigkeit zusammen. Das charakteristische

System ist vollständig integrabel, die Funktionen F_ρ sind Integrale des charakteristischen Systems. Unter diesen Voraussetzungen beweist Verfasser die Lösbarkeit des Cauchyschen Anfangswertproblems und behandelt die Beispiele

$$p_1^2+(x_1-x_2)p_3=0,\ p_1+p_2=0$$

mit den Anfangsbedingungen

$$x_1=s_1,\ x_2=-s_1,\ x_3=0,\ u=s_1$$

und

$$p_1p_2=x_1x_2,\ p_3p_4=x_3x_4$$

mit den Anfangsbedingungen

$$x_1=x_2,\ x_3=x_4,\ u=x_1^2+x_2^2.$$

Schlieszlich wird auch der Fall untersucht, dasz die unbekannte Funktion u explizit im vorgelegten Differentialsystem auftritt. *M. Pinl* (Köln).

Smirnov, S. V. Cauchy's problem for a system of linear partial differential equations. Ivanov. Gos. Ped. Inst. Uč. Zap. Fiz.-Mat. Fak. 1 (1941), no. 1, 36–41. (Russian)

The author considers the Cauchy problem consisting of the system

$$\frac{\partial u_i}{\partial t}=\sum_{k=1}^{2v-1}\sum_{j=1}^{n}a_{ij}{}^k\frac{\partial u_j}{\partial x_k}+\sum_{j=1}^{n}b_{ij}u_j+\varphi_i\ (i=1,2,\cdots,n),$$

with $a_{ij}{}^k$, b_{ij}, φ_i functions of $(x_1, x_2, \cdots, x_{2v-1}, t)$, subject to the initial conditions

$$u_i(x_1,x_2,\cdots,x_{2v-1},0)=\bar{u}_i(x_1,x_2,\cdots,x_{2v-1})\quad(i=1,2,\cdots,n),$$

where the $\bar{u}_i$ are given functions. Assuming the existence of a certain "Green's matrix", this Cauchy problem is reduced to an equivalent integral equation [cf. S. Sobolev, Mat. Sb. N.S. 1(43) (1936), 39–72]. The actual construction of the Green's matrix is to be taken up elsewhere. *J. B. Diaz* (Cambridge, Mass.).

Fulks, W. A note on the steady state solutions of the heat equation. Proc. Amer. Math. Soc. 7 (1956), 766–770.

Let R be an open connected set in the $x=(x_1, \cdots, x_n)$ space and let B denote the boundary of R. This paper deals with two results of Tychonoff [Bull. Univ. d'Etat Moscou. Sér. Internat. Sect. A. 1 (1938), no. 9, 1–44]. The first concerns solutions $u=u(x)$ of the Laplace equation $\Delta u=0$ as limits, as $t\to\infty$, of solutions $u=u(x, t)$ of the heat equation $\Delta u=u_t$; the second states that if R is a fundamental domain for $\Delta u=u_t$ (that is, if $\Delta u=u_t$ has a unique continuous solution for any given continuous Dirichlet data), then it is a fundamental domain for $\Delta u=0$. Fulks gives proofs of these results. He reverses Tychonoff's procedure by proving the second theorem first by showing that if $x=x^0\in B$, then certain solutions of $\Delta u=u_t$ are barriers at $x=x^0$. *P. Hartman.*

Bergman, Stefan. Bounds for solutions of a system of partial differential equations. J. Rational Mech. Anal. 5 (1956), 993–1002.

For ψ satisfying a system of two second order elliptic equations (in complex notation)

$$(\partial^2\psi/\partial z_k\partial\bar{z}_k)+F_k(z_k,\bar{z}_k)\psi=0\ (k=1,2).$$

one can obtain a certain bound on $|\psi(z_1,\bar{z}_1,z_2,\bar{z}_2)|$ valid in $[|z_1|\leq 1]\times[|z_2|\leq 1]$. The method of proof depends upon the use of integral operators to map pairs of analytic functions (g_1, g_2) of two complex variables into solutions ψ; the proof is carried out under the assumption that $g_1(0, 0)$ is real, but can be modified to cover the case where this is not true. The method also extends to nonlinear systems of the form

$$(\partial^2\phi/\partial z_k\partial\bar{z}_k)+F_k(z_k,\bar{z}_k)\phi=P_k(\phi,z_1,\bar{z}_1,z_2,\bar{z}_2)\ (k=1,2),$$

where the F_k are nonpositive and the P_k are nonnegative. *R. B. Davis* (Syracuse, N.Y.).

Bers, Lipman. Survey of local properties of solutions of elliptic partial differential equations. Comm. Pure Appl. Math. 9 (1956), 339–350.

An expository article describing known results and some open questions dealing with solutions of elliptic partial differential equations. In particular, the article is concerned with problems of local smoothness, behavior near a zero and near a singularity, asymptotic and convergent expansions at a point, etc. There is an extensive bibliography. *P. Hartman* (Baltimore, Md.).

★ Tautz, G. L. Zur Theorie des Dirichletschen Problems. Convegno Internazionale sulle Equazioni Lineari alle Derivate Parziali, Trieste, 1954, pp. 97–102. Edizioni Cremonese, Roma, 1955. 3000 Lire.

Serrin, James. On the Harnack inequality for linear elliptic equations. J. Analyse Math. 4 (1955/56), 292–308.

The inequality referred to is the well known result that there exists a constant K such that if $u(P)$ is a function which is positive and harmonic in the sphere $0\leq|P|<1$, then

$$\frac{1}{K}u(Q)\leq u(0)\leq Ku(Q)$$

whenever $0\leq|Q|<\frac{1}{2}$. The author extends this theorem to positive solutions of elliptic equations of the form

$$(1)\qquad Lu=a_{ij}(P)u_{ij}+b_i(P)u_i+c(P)u=0\quad(c\leq 0),$$

where

$$u_i=\frac{\partial u}{\partial x_i},\ a_{ij}u_{ij}=\sum_{i,j=1}^{n}a_{ij}\frac{\partial^2u}{\partial x_i\partial x_j},\ \text{etc.}$$

Results of this type have been obtained previously by Lichtenstein [Rend. Circ. Mat. Palermo 33 (1912), 201–211] and by Feller [Math. Ann. 102 (1930), 633–649]; the significant feature of the present work is that relatively little is assumed about the smoothness of the coefficients, thus suggesting the possibility of applications to nonlinear problems. Thus, for $n=2$ it is supposed only that the coefficients are bounded and that (1) is uniformly elliptic; for $n>2$ the requirement is introduced that the a_{ij} are continuous on the surface $|P|=1$. For $n=2$ a proof under similar hypotheses has been given by Bers and Nirenberg [Convegno Internazionale sulle Equazioni Lineari alle Derivate Parziali, Trieste, 1954, Edizioni Cremonese, Roma, 1955, pp. 141–167; MR 17, 974]; the proof of the author has the advantage that it is elementary, being based only on the maximum principle and on a comparison device due to E. Hopf [S.-B. Preuss. Akad. Wiss. 1927, 147–152]. In the case $n>2$ the author constructs a function with properties analogous to those of the Poisson kernel, and the proof then proceeds as for harmonic functions.

As an application, the author proves that if $u(P)$ is a solution over the whole plane of a uniformly elliptic equation

$$Au_{xx}+2Bu_{xy}+Cu_{yy}=0 \tag{2}$$

and if u is bounded from below, then $u\equiv$const. This complements a theorem of S. Bernstein [Math. Z. 26 (1927), 551–558] on bounded solutions of (not necessarily uniformly) elliptic equations (2), and a theorem of the reviewer [C. R. Acad. Sci. Paris 235 (1952), 596–598; MR 14, 364] on solutions, bounded below, of equations in a class which overlaps (2).

The author shows that for $n>2$ the assumption of continuity of the coefficients can be replaced by an assumed bound on the gradient of the solution. He gives also an extension of his result to an inhomogeneous equation (*). *R. Finn* (Pasadena, Calif.).

Gilbarg, D.; and Serrin, James. On isolated singularities of solutions of second order elliptic differential equations. J. Analyse Math. 4 (1955/56), 309–340.

This paper presents an extensive and detailed discussion of the behavior of solutions of second order elliptic partial differential equations

$$Lu=a_{ik}(x)u_{ik}+b_i(x)u_i+c(x)u=f(x),$$

$$x=(x_1,\cdots,x_n),\ u_i=\frac{\partial u}{\partial x_i},\ b_iu_i=\sum_{i=1}^n b_iu_i,\ \text{etc.}, \tag{1}$$

which are defined in a neighborhood of an isolated singular point P and subject to a growth condition similar to that of the fundamental singularity associated with (1). The authors show that the possible behavior of a solution depends critically on the behavior of the coefficients a_{ik} at P. Thus, if the a_{ik} are Dini continuous at P, if $c\equiv 0$ and $Lu\geqq 0$, and

$$u=\begin{cases} o(\log r), & n=2,\\ o(r^{2-n}), & n>2,\end{cases}$$

then u satisfies an extended maximum principle at P. If the a_{ik} are merely continuous this result is in general false, as the authors show by example. They prove, however, that for $n>2$ the Dini continuity of the a_{ik} can be replaced by the strengthened growth condition $u=O(r^{2-n+\delta})$ for any $\delta>0$. The authors then extend this result to show that $u(x)$ tends to a limit at P (for $n=2$ this is true even for discontinuous coefficients, under the single assumption $u\geqq -M$). As a corollary they obtain analogues of Liouville's theorem for solutions of $a_{ik}u_{ik}=0$ which are defined over all space. The authors also present theorems on the unique characterization of positive singularities. Proofs are based on a comparison device due to E. Hopf, and on a Harnack-type inequality of J. Serrin [cf. the preceding review].

The paper includes several analogues of the above results for equations with a divergence structure,

$$(a_{ik}u_i)_k+b_iu_i=0$$

for which the ellipticity is permitted to break down in certain ways at the singular point. In this case a somewhat different relation is seen to hold between smoothness conditions on the coefficients and behavior of the solutions.

The connection between the behavior of the coefficients and that of the solutions is clarified with two instructive examples. *R. Finn* (Pasadena, Calif.).

Foguel, Shaul Reuven. A note on a problem related to the Dirichlet problem. Riveon Lematematika 9 (1955), 18–22. (Hebrew. English summary)

Let Ω be a bounded domain in three-space with boundary F. Given a continuous function U on F, one can always find a function V which is harmonic in Ω, and has on F the boundary values U except for the set of singular points on F which is independent of U and has the capacity zero. The author studies the question under what conditions a singular point on F can be regular with respect to a particular function U on F. By elementary considerations, he finds the necessary and sufficient condition: $\int_F U d\mu=U(M)$, where μ is the "surbalayage polaire" defined by de La Vallée Poussin [Le potentiel logarithmique, Louvain, 1949]. *M. Schiffer.*

Supino, G. Limitazioni e confronti per le funzioni metaarmoniche. Rev. Un. Mat. Argentina 17 (1955), 287–292 (1956).

The author reviews some of the known properties of the solutions of the elliptic equation $\Delta u-\lambda^2u=0$. For example maximum and minimum properties of such solutions in a bounded plane region D are discussed as well as the rate of growth of their derivatives on approach to the boundary of D. *F. G. Dressel.*

Blagoveščenskiĭ, Yu. V.; and Fil'čakov, P. F. Solution of plane problems of torsion and bending by the method of electrohydrodynamical analogies. Akad. Nauk Ukrain. RSR. Prikl. Meh. 1 (1955), 195–204. (Ukrainian. Russian summary)

An approximate method of solution of the Dirichlet problem is described as a sequence of solutions of the problem for a circle. Given an arbitrary, simply-connected region D, whose boundary is composed of a finite number of arcs each with continuously turning tangent. Circumscribe a circle K about D, which may coincide with the boundary of D at some points. Choose the values of the potential ϕ on the boundary of K so that as nearly as can be determined the resulting values of ϕ on the boundary of D will be as close as possible to the given values. Solving the Dirichlet problem for K then gives an approximate solution. ϕ_0, for D. A Fourier series expansion for ϕ is then used to find the values of ϕ_0 on the boundary of D. Define the error function ϕ_1 as the difference between ϕ_0 and the exact values of ϕ on D. Then ϕ_1 is harmonic in D. Choose the boundary values of ϕ_1 on D as the difference between the computed values of ϕ_0 and the values given originally. Thus ϕ_1 satisfies a Dirichlet problem in D. Again solve this problem for the circumscribed circle K in the same manner as above. This process may be repeated as often as necessary to obtain a desired degree of approximation.

Results are indicated for the torsion problem of an equilateral triangle where the boundary values of the potential satisfy $\phi=\frac{1}{2}r^2$, where r is the radius vector. The second approximation is shown to be sufficient for this problem.

Reference is made to an electrodynamical analogue computer which can be used to solve the potential problem. *H. P. Thielman and H. J. Weiss.*

Courant, R.; and Lax, P. D. The propagation of discontinuities in wave motion. Proc. Nat. Acad. Sci. U.S.A. 42 (1956), 872–876.

The authors prove the following theorem: Let

$$Mu=\sum_{\nu=0}^n A^\nu(\partial u/\partial x^\nu)+Bu=0$$

be a hyperbolic system of k first-order linear equations in k unknowns, in which A^0 is the identity matrix and all linear combinations with real coefficients of the matrices A^ν have real and distinct eigenvalues. Let initial data $u(0, x^1, \cdots, x^n)=f(x^1, \cdots, x^n)$, having continuous derivatives of sufficiently high order on either side of a sufficiently smooth $(n-1)$-dimensional manifold Γ which suffer jump discontinuities across Γ, be prescribed on $x^0=0$. Then the (weak) solution $u(x^0, x^1, \cdots, x^n)$ of this initial-value problem has continuous partial derivatives of sufficiently high order everywhere except on the k characteristic surfaces issuing from Γ; across these the partial derivarives have jump discontinuities. The authors show also that the differentiability properties of a solution at a point P at time T depend only on the differentiability properties of the initial data in the neighborhood of those points which lie on the bicharacteristics issuing from (T, P). This last result they call a generalized Huygens' principle. *R. N. Goss.*

Blondel, Jean-Marie. Sur le comportement à l'infini des solutions d'une équation linéaire aux dérivées partielles du second ordre. C. R. Acad. Sci. Paris **243** (1956), 833–835.

Soit $z(x, y)$ solution de $\partial^2 z/\partial x\partial y+A(x, y)z=0$, $z(x, 0)=\varphi(x)$, $z(0, y)=\psi(y)$, $x, y\geqq 0$. Résultats sur le comportement de $z(x, y)$ lorsque $x\to+\infty$, $y\in$ compact. Par exemple, si $\partial A/\partial x\leqq 0$, si $|\varphi'|$ est bornée et $L^2(0, +\infty)$, ψ continue, alors $|\partial z/\partial x|$ est bornée pour $x\geqq 0$, $y\in$ compact.

J. L. Lions (Nancy).

Halilov, Z. I. On the theory of a method of solution of mixed problems. Dokl. Akad. Nauk Azerbaĭdžan. SSR. **9** (1953), 425–429. (Russian. Azerbaijani summary)

The following problem is studied:

$$(1)\quad \frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}-C(x, t)u;\quad u|_{x=0}=0,\ 0\leqq t\leqq T;\quad u|_{t=0}=f(x),\ 0\leqq x<\infty.$$

One seeks the solution in the form

$$u(x, t)=\int_0^\infty A(t, \lambda)\sin\lambda x\, d\lambda,$$

where

$$A(t, \lambda)=\frac{1}{\lambda^2}\int_0^t e^{-\lambda^2(t-\tau)}\varphi(\tau, \lambda)d\tau+a(\lambda)$$

and

$$a(\lambda)=\frac{2}{\pi}\int_0^\infty f(x)\sin\lambda x\, dx.$$

For the function $\varphi(t, \lambda)$ we obtain the integral equation

$$(2)\quad \varphi(t, \lambda)=\int_0^t d\tau\int_0^\infty K(t, \lambda; \tau, \mu)\varphi(\tau, \mu)d\mu+\psi(t, \lambda),$$

with

$$K(t, \lambda; \tau, \mu)=\frac{\lambda^2}{\mu^2}B(t, \lambda; \mu)e^{-\mu^2(t-\tau)},$$

$$\psi(t, \lambda)=\lambda^2\int_0^\infty B(t, \lambda; \mu)a(\mu)d\mu-\lambda^4 a(\lambda),$$

$$B(t, \lambda; \mu)=-\frac{2}{\pi}\int_0^\infty C(x, t)\sin\lambda x\sin\mu x\, dx.$$

Then equation (2) is investigated.

The author remarks that his method can be extended to linear problems of higher order with an arbitrary number of variables. A detailed presentation is to appear in a special paper of the author.

B. M. Levitan (RŽ Mat. No. 2152, 1954).

Lauwerier, H. A. Diffusion from a point source into a space bounded by an impenetrable plane. Appl. Sci. Res. A. **6** (1956), 197–204.

The paper gives a solution for the diffusion equation in half-space with cylindrical symmetry for the case when a source is placed at the origin. The physical problem is that of finding the concentration of particles of a metal evaporated between two electrodes. The plane $z=0$ is assumed reflecting (zero mass transport). The solution is obtained by the method of sources (placed along z-negative). *J. Kestin* (Providence, R.I.).

Beattie, I. R.; and Davies, D. R. A solution of the diffusion equation for isotopic exchange between a semi-infinite solid and a well stirred solution. Phil. Mag. (8) **1** (1956), 874–879.

The problem of the diffusion of material from a semi-infinite solid of known surface area into a given volume of well-stirred liquid is equivalent to solving

$$\frac{\partial C}{\partial t}=D\frac{\partial^2 C}{\partial x^2}\ (x>0, t>0)$$

where D is a constant, under the conditions

$$C=C_0\ \text{(constant)}\quad (x>0, t=0),$$
$$C=0\quad (x=0, t=0),$$
$$\frac{\partial C}{\partial x}=\frac{V}{DA}\frac{\partial C}{\partial t}\quad (x=0, t>0),$$

where V/A is a constant. This problem is solved by means of a Laplace transform with respect to t. *E. T. Copson.*

Sagomonyan, A. Ya. Reflection of a shock wave from the vertex of a cone and the problem of its penetration into a compressible fluid. Vestnik Moskov. Univ. **11** (1956), no. 3, 3–17. (Russian)

Consider the unsteady axisymmetric flow produced when a plane shock impinges upon a circular cone. The author assumes that the cone is so blunt that regular reflection occurs at the intersection of the cone and main shock, and also that entropy variations are negligible, so the flow is approximately irrotational. Then the velocity potential is of the form $\bar\varphi(x_0, y_0, t)=t\varphi x_0/t, y_0/t)$. In the region of diffracted flow near the vertex of the cone the partial differential equation satisfied by φ is of elliptic type; in the region between the cone and the regularly reflected part of the shock it is hyperbolic. For the non-linearized partial differential equation the author exhibits characteristic equations, boundary conditions, and presents results of a numerical calculation of the shock location at the edge of the hyperbolic region of unsteady flow. For the linearized equation, while retaining the exact boundary conditions at the cone and by keeping the tangential velocity component continuous at the reflected "weak shock" (actually a characteristic) he is able to obtain explicit formulas for velocity components and pressure on the reflected shock. For very blunt cones, for which some boundary conditions are applied on the plane $x_0=0$ rather than on the cone, he finds the velocity potential by means of Kirchoff's

formula for the wave equation. He compares computations of velocity and pressure distributions obtained by both linearized methods for a cone with 75° semi-vertex angle. He also sketches a parallel treatment of the entry of a cone into a compressible fluid with a plane free surface.

J. H. Giese (Aberdeen, Md.).

★**Schlögl, F. Randwertprobleme.** Handbuch der Physik. Bd. I. Mathematische Methoden I, pp. 218–352. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1956. DM 72.00.

This useful survey report covers a multitude of subjects, which it is not possible to discuss here in detail. There are four main sections: A. Orthogonal systems of functions, B. Linear integral equations, C. Calculus of variations D. Boundary value problems for equations of mathematical physics. Section A covers expansions in series of orthogonal functions, Fourier series, and linear transformations in function space. Section B deals with both hermitian and arbitrary kernels, and direct methods of solution. The first three sections are, in a sense, introductory to section D, which contains such topics as linear second order equations, Green's function, eigenvalue problems, connection with the calculus of variations, and others. The discussion of the Klein-Gordon equation:

$$\Delta\psi-\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}-x^2\psi=f,$$

is of particular interest. *J. B. Diaz.*

★**Riesz, Marcel. Problems related to characteristic surfaces.** Proceedings of the conference on differential equations (dedicated to A. Weinstein), pp. 57–71. University of Maryland Book Store, College Park, Md., 1956.

In 1949, Professor Riesz published a long paper in which he introduced a multiple integral of Riemann-Liouville type and showed how important this idea is in the theory of the wave equation. [Acta Math. **81** (1949), 1–223; MR **10**, 713]. By this means he gave the solution of the problem of Cauchy for a space-like boundary and also the solution of a characteristic boundary value problem when the unknown function is given on a characteristic half-cone.

He now gives the solution of the wave equation for a very general class of characteristic boundaries. In the three-dimensional case, a specialization of the solution leads to an interesting case of curvatura integra for twisted curves. The differential-geometrical consequences of the formula obtained are examined independently.

It is known that discontinuities of a solution of a partial differential equation can occur only on characteristic surfaces of the equation. In the last section a sequence of special functions is constructed which are related to a given characteristic surface in the same way as the powers of the Lorentz distance are related to a characteristic cone. In particular, the author gives an explicit solution of the wave equation which is infinite on the characteristic surface in the same way as the elementary solution is on a characteristic cone. *E. T. Copson.*

Bergman, Stefan. New methods for solving boundary value problems. Z. Angew. Math. Mech. **36** (1956), 182–191. (German, French and Russian summaries)

In this expository paper, some methods are considered which yield sets of particular solutions for linear partial differential equations. Bergman's integral operators of the first and third kind are described which apply to second order equations in two variables with analytic coefficients. Extensions to problems in three variables are indicated and numerous references to papers by the author and his collaborators are provided. Finally, the value of particular solutions is discussed in the case of two-dimensional compressible fluid flows. In order to facilitate the orthogonalization of solutions which is necessary for solving the boundary value problem, the author proposes to approximate arbitrary domains in the pseudo-graph plane by domes with horizontal and vertical boundaries only [see Bergman, J. Rational Mech. Anal. **4** (1955), 883–905; MR **17**, 549]. *M. Schiffer.*

Tautz, Georg Lukas. Zur Theorie des Dirichletschen Problems bei nichtlinearen elliptischen Differentialgleichungen. Rend. Sem. Mat. Univ. Padova **24** (1955), 421–442.

In previous papers [Arch. Math. **3** (1952), 232–238, 239–250, 361–365; MR **14**, 876, 877], the author has investigated linear spaces of functions defined in some domain of Euclidean space with the property: if two members of such a family agree on the boundary of a subdomain, they agree throughout the subdomain, and furthermore the manner of dependence of functions of this space on their boundary values is like the dependence of solutions of second order elliptic equations on their boundary values. The author has shown that under these hypotheses (stated there precisely) this linear space of functions is the nullspace of some elliptic operator. In the present paper the author presents an intrinsic characterisation of families of functions which are solutions of a nonlinear second order elliptic equation. *P. D. Lax.*

Kalitzin, Nikola St. Taylor'sche Reihen und Randwertaufgaben. Univ. d'État Varna "Kiril Slavianobâlgarski" Fac. Tech. Méc. Annuaire **3** (1947–1948), 195–243 (1949). (Bulgarian. German summary)

Adler, F. T.; and Baroncini, D. Approximations for linear betatron oscillations. Nuovo Cimento (10) **4** (1956), 959–974.

Approximation methods for calculating the characteristic exponent of extended Hill equations are derived and applied to the computation of linear betatron oscillations. *Author's summary.*

Alaci, V. Equations aux derivées partielles et aux coefficients variables du 3^e et 4^e ordre. Acad. R. P. Romîne. Baza Cerc. Şti. Timişoara. Stud. Cerc. Şti. Ser. I. **2** (1955), 9–12. (Romanian. Russian and French summaries)

Consider the equation

$$(*)\qquad \sum_{i=1}^{n}\frac{1}{a_i^2x_i}\frac{\partial^3u}{\partial x_i^3}=0,$$

with constant a_i's and set $v=\sum_{i=1}^n a_ix_i^2$. Then $u=\varphi(v)$ is a particular solution of (*), provided that $\varphi(v)$ satisfies the Euler type equation $2v^3\varphi'''+3nv^2\varphi''=0$. Hence, one obtains particular solutions of (*), depending on arbitrary constants. Similarly,

$$\sum_{i=1}^{n}\frac{1}{a_i^3x_i^3}\left(x_i\frac{\partial^4u}{\partial x_i^4}-\frac{\partial^3u}{\partial x_i^3}\right)=0$$

admits the solution $u=\varphi(v)$, provided that φ satisfies the

Euler-type equation $2v^4\varphi^{(4)}+5nv^3\varphi'''=0$ and the procedure may be generalized to equations with partial derivatives of higher order. *E. Grosswald.*

Payne, L. E. New isoperimetric inequalities for eigenvalues and other physical quantities. Comm. Pure Appl. Math. **9** (1956), 531–542.

This paper is a highly concentrated collection of new inequalities relating the eigenvalues of various physical problems connected with the Laplacian and the bi-Laplacian operators.

For a fixed two-dimensional domain the author considers the torsional rigidity P, the torsion function v, the membrane eigenvalues λ_n, the buckling eigenvalues Λ_n, the vibrating plate eigenvalues Ω_n as well as buckling eigenvalues of a vibrating plate and vibration eigenvalues of a plate under compression. An example of the inequalities between these quantities is

$$P\geqq 2\pi v_{\max}>8\pi/\lambda_1^2.$$

Other inequalities give upper bounds for the first two Stekloff eigenvalues for the exterior of a closed surface C^* in three dimensions in terms of the electrostatic capacity, surface area, and virtual mass of C^*. If V is the volume enclosed by C^*, and P_{ave} and W_{ave} its average polarization and virtual mass, respectively, the author shows that if C^* is axially symmetric

$$P_{\text{ave}}\geqq 2V.$$

If, in addition, C^* is not "ring shaped",

$$W_{\text{ave}}\geqq \tfrac{1}{2}V,$$

which was conjectured for general C^* by Pólya [Proc. Nat. Acad. Sci. U.S.A. **33** (1947), 218–221; MR **9**, 111].

H. F. Weinberger (College Park, Md.).

Il'in, V. A. Proof that a function with a singularity can be expanded into a series of characteristic functions. Dokl. Akad. Nauk SSSR (N.S.) **109** (1956), 21–24. (Russian)

The author proves that if $v(x)=v(x_1,\cdots,x_n)$ can be represented by a series of eigenfunctions of $\Delta u+\lambda u=0$ in a given domain g, and if $\phi(x)=cr^{-\alpha}+v(x)$, $0<\alpha<1$ (or $\phi(x)=c\log r+v(x)$), where $r=(\sum_1^n x_i^2)^{1/2}$, then $\phi(x)$ also admits such a representation. The proof is based on a representation, for a particular function with the prescribed singularity, obtained by the author in a previous work [Dokl. Akad. Nauk. SSSR (N.S.), **105** (1955), 18–21; MR **17**, 744]. The author proves the uniform and absolute convergence of the representation in any compact subregion which excludes the singular point.

R. Finn (Pasadena, Calif.).

See also: Kamke, p. 384; Il'in, p. 392; Bochner, p. 404; Heinz, p. 413; Brazma, p. 419; Blanc, p. 420; Patlak, p. 424; Sack, p. 429; Vorovič, p. 434; Chakravorty, p. 434; Chilver, p. 436.

Difference Equations, Functional Equations

Polosuhina, O. A. On the solution of finite-difference equations in several independent variables. Leningrad. Gos. Univ. Uč. Zap. 144. Ser. Mat. Nauk **23** (1952), 111–150. (Russian)

"In the present paper are considered several methods of solving linear equations with partial differences, i.e. linear difference equations with several independent variables, considered under the assumption that the independent variables vary continuously, rather than from the point of view of recurrence series, when the independent variable takes on only integral values. The methods given here are close to the methods of solution of ordinary difference equations." (From the author's summary.) *J. M. Danskin* (Princeton, N.J.).

Hahn, Wolfgang. Bemerkungen zu der Arbeit über Differential-Differenzengleichungen Bd. 131, S. 151 (1956). Math. Ann. **132** (1956), 94.

The author discusses some modifications and corrections in his paper cited in the title [see MR **17**, 1215].

R. Bellman (Santa Monica, Calif.).

See also: Nikolaeva, p. 385; Mikeladze, p. 419; Kunz, p. 419.

Calculus of Variations

Young, L. C. A variational algorithm. Riv. Mat. Univ. Parma **5** (1954), 255–268.

The author terms the "homology principle" of the calculus of variations the existence of a non-negative integrand $f+\phi$ whose integral vanishes on some admissible locus, the desired minimizing curve or surface, and to which the integrand f of the given problem is homologous in the following sense. Denote by (L,f) the integral of f on the surface L. That $f+\phi$ is homologous to f means that (L,ϕ) depends only on the boundary of L. If now some non-negative $f+\phi$ is homologous to f and $(L_0,f+\phi)=0$ for some surface L_0, then if L is any other surface with the same boundary as L_0 one has $(L,\phi)=(L_0,\phi)$ and therefore

$$(L,f)-(L_0,f)=(L,f+\phi)-(L_0,f+\phi)=(L,f+\phi)\geqq 0,$$

so that the minimum of (L,f) is attained for $L=L_0$ in the class of L with the same boundary as L_0 with no restriction on the topological type of L.

The author proves a converse to the homology principle, after admitting certain discontinuous integrands. In fact, if L_0 is a generalized surface in the sense of Young with g-boundary λ_0, which latter is "polyhedrally approximable" in a certain sense, such that $(L_0,f)\leqq(L,f)$ for all generalized surfaces L with the same g-boundary λ_0, then there exists a non-negative f_1, homologous to f_0, such that $(L_0,f_1)=0$. *J. M. Danskin.*

See also: Young, p. 384; Ulucay, p. 385; Schlögl, p. 401; Leepin, p. 425; Colonnetti, p. 431; Koppe, p. 431; Nudel'man and Ovčinnikov, p. 432; McComb, p. 434; Horváth, p. 441; Dalgarno and Lewis, p. 443; Szamosi, p. 445.

TOPOLOGICAL ALGEBRAIC STRUCTURES

Topological Groups

Kneser, Martin. Summenmengen in lokalkompakten abelschen Gruppen. Math. Z. 66 (1956), 88–110.

The author extends results of Lusternik [C. R. (Dokl.) Acad. Sci. URSS (N.S.) 3 (1935), 55–58], Raikov [Mat. Sb. N.S. 5(47) (1939), 425–440; MR 1, 296], Henstock and Macbeath [Proc. London Math. Soc. (3) 3 (1953), 182–194; MR 15, 109], Macbeath [Proc. Cambridge Philos. Soc. 49 (1953); 40–43; MR 15, 110] and the reviewer [Fund. Math. 42 (1955), 57–60; MR 17, 245], on the measure of sum sets. In an abelian topological group $A+B$ denotes the set of all sums $a+b$, $a\in A$, $b\in B$; μ_* is the inner Haar measure.

Theorem 1. If G is a locally compact abelian group, then $\mu_*(A+B)\geqq\mu(A)+\mu(B)$, unless there is an open compact subgroup H such that: (i) $A+B+H=A+B$, and (ii) $\mu(A+B+H)=\mu(A+H)+\mu(B+H)-\mu(H)$. In this case one has: $\mu(A+B)\geqq\mu(A)+\mu(B)-\mu(H)$. Corollary. If G is compact and connected then (1) $\mu_*(A+B)\geqq\mu(A)+\mu(B)$ if $\mu(A)+\mu(B)\leqq\mu(G)$, and $A+B=G$ otherwise.

Theorem 2. If G is compact and connected and if $\mu(A)$, $\mu(B)>0$, $\mu(A)+\mu(B)<\mu(G)$, then equality holds in (1) if and only if there is a character ϕ of G, and two intervals K, L, in the circle group such that $A\subset\phi^{-1}(K)$, $B\subset\phi^{-1}(L)$, $\mu(A)=\mu(\phi^{-1}(K))$, $\mu(B)=\mu(\phi^{-1}(L))$. Conditions are also given for equality to hold in (1) in case one set has measure zero. The case when both sets have measure zero seems to be very difficult.

Theorem 4. If $G=F\oplus R^n$, where F is locally compact and has a open compact subgroup, and R^n is the real n-dimensional Euclidean space, then

$$\mu_*(A+B)^{1/n}\geqq\mu(A)^{1/n}+\mu(B)^{1/n}.$$

Here too conditions for equality are given in case at least one set has positive measure.

The author concludes with an interesting generalization of the concept of asymptotic density of a sequence of positive integers. He considers an abelian subsemigroup E that is dense in a compact group G. It is assumed that a "density" function δ is defined for subsets of E. A J-set (Jordan measurable) in G is a set A such that $\mu(A^0)=\mu(\bar{A})$, where A^0 is the interior of A. For J-sets one has: $\delta(A\cap E)=\mu(A)$. Sets of the form $A\cap E$, where A is a J-set are called almost periodic subsets of E. For these sets the analogue of the $\alpha+\beta$ theorem holds:

$$\delta(C+D)\geqq\delta(C)+\delta(D),$$

with certain exceptions analogous to those of Theorem 1.

A. Shields (Ann Arbor, Mich.).

Balcerzyk, S.; and Mycielski, Jan. On free subgroups of topological groups. Bull. Acad. Polon. Sci. Cl. III. 4 (1956), 415.

The paper announces the following results: A free group of rank $2^{\aleph_0}$ appears as a subgroup in 1) every compact, connected, non-abelian group, and 2) every locally compact, connected non-solvable group. A free abelian group of rank $2^{\aleph_0}$ appears in every locally compact, non-zero dimensional, abelian group.

A. M. Gleason (Cambridge, Mass.).

Walker, Elbert A. Cancellation in direct sums of groups. Proc. Amer. Math. Soc. 7 (1956), 898–902.

Let F, G and H be groups. Suppose that $F+G$ (direct sum) is isomorphic to $F+H$. Can we conclude that G and H are isomorphic? 1) Yes, if F is finite. In a footnote it is observed that this appears in Jónsson and Tarski, "Direct decomposition of finite algebraic systems" [Univ. of Notre Dame, 1947; MR 8, 560]. 2) Yes, if G and H are abelian and F is finitely generated. This answers the reviewer's "Test Problem III" [Infinite abelian groups, Univ of Michigan Press, 1954; MR 16, 444]. An independent proof has been given by P. Cohn [Proc. Amer. Math. Soc. 7 (1956), 520–521; MR 17, 1182]. 3) No, in the non-abelian case, even if F is infinite cyclic. The author attributes the example to W. R. Scott. A number of related results are proved.

I. Kaplansky.

See also: Cotlar, p. 383; Tsuji, p. 389; Kim Sen En and Morozov, p. 403; Rhodes, p. 406.

Lie Groups and Algebras

Stoka, Marius I. Sur les groupes G_r mesurables d'un espace R_n. Com. Acad. R. P. Romîne 6 (1956), 745–751. (Romanian. Russian and French summaries)

Let G_r be a Lie transformation group operating in E_n. It is shown that the group admits an invariant integral in E_n if and only if it operates transitively and certain relations involving the infinitesimal transformations and their derivatives are satisfied. If $r=n$, transitivity alone is necessary and sufficient.

P. A. Smith.

Kim Sen En; and Morozov, V. V. On imprimitive groups of the three-dimensional complex space. Uč. Zap. Kazan. Univ. 115 (1955), no. 14, 69–85. (Russian)

In elaboration, extension and improvement of results announced earlier by the first author [Dokl. Akad. Nauk SSSR (N.S.) 99 (1954), 205–207; MR 16, 567] the authors derive the structure of the various possible imprimitive groups of transformations in complex three space in the special case in which the corresponding abstract group is not solvable. In the more complicated solvable case they confine their attention to the intransitive groups and to one class of transitive groups. They also rederive the known results for two space. All considerations are local.

G. W. Mackey (Cambridge, Mass.).

See also: Patterson, p. 375; Higman, p. 377; Séminaire Henri Cartan, p. 409.

Topological Vector Spaces

Słowikowski, W.; and Zawadowski, W. Note on relatively complete B_0-spaces. Studia Math. 15 (1956), 267–272.

A B_0-space L is a complete linear metric space in which the topology is defined by a sequence of pseudonorms $|x|_k$. Let L_k be the closed linear subspace $\{x\colon x\in L$ and $|x|_k=0\}$ L is called relatively complete if the sequence of pseudonorms defining the topology can be chosen so that each L/L_k is a complete normed space. Set $|x|_{nk}=\inf\{|x-y|_n\colon y\in L_k\}$. The authors show first that the space of functions analytic in the unit circle is a complete B_0-space but is not relatively complete. The rest of the paper is devoted to proof of: A B_0-space L is relatively complete

if and only if there is a double sequence a_{nk} of positive numbers such that for each k and for each x in L $\sup_n a_{nk}|x|_{nk}<\infty$. *M. M. Day* (Seattle, Wash.).

Klee, V. L., Jr. Strict separation of convex sets. Proc. Amer. Math. Soc. **7** (1956), 735–737.

In a general way the significant closure types of properties for convex sets are in terms of intersections with rays whence in particular (a) the boundedness of such sections ought be necessary for compactness in natural topologies. Separation theorems for convex sets usually involve compactness conditions. This paper establishes that in a topological linear space locally compact closed convex sets satisfying (a) are compact. This leads to results implying in particular that strict separation of disjunct closed convex sets in an n dimensional Euclidean space is equivalent to the non-existence of rays in their boundaries. *D. G. Bourgin* (Urbana, Ill.).

Cristescu, Romulus. L'opérateur-produit dans les espaces linéaires semi-ordonnés et ses applications. Acad. R. P. Romîne. Stud. Cerc. Mat. **6** (1955), 357–493. (Romanian. Russian and French summaries)

Let $\mathscr{X}$, $\mathscr{Y}$, and $\mathscr{Z}$ be vector lattices. Let $\langle x, y\rangle$ be an operation with domain contained in the Cartesian product $\mathscr{X}\times\mathscr{Y}$ and range contained in $\mathscr{Z}$, having the following properties. The equality $\langle x_1+x_2, y\rangle=\langle x_1, y\rangle+\langle x_2, y\rangle$ holds whenever the right side is defined. The equality $\langle x, y_1+y_2\rangle=\langle x, y_1\rangle+\langle x, y_2\rangle$ holds whenever the right side is defined. If $x\geqq 0$, $y\geqq 0$, and $\langle x, y\rangle$ exists, then $\langle x, y\rangle\geqq 0$. If $\langle x_1, y_1\rangle$ exists and $|x|\leqq|x_1|$, $|y|\leqq|y_1|$, then $\langle x, y\rangle$ exists. Also $\langle 0, y\rangle$ and $\langle x, 0\rangle$ exist for all $x, y \in \mathscr{X}\times\mathscr{Y}$. The present memoir is a detailed survey of the theory of these operator products. The general form of, and methods of constructing, operator products with certain additional properties of continuity and disjunctivity are given. Solutions of equations of the form $y-\langle x_0, y\rangle=y_0$ are given for a number of cases (here one must have $\mathscr{Y}\subset\mathscr{Z}$.)

The second half of the memoir deals with a certain theory of integration. A σ-complete vector lattice $\mathscr{X}$ is said to be of type R^* if for every double sequence $\{x_i^k\}_{i,k=1}^{\infty}$ from $\mathscr{X}$ such that $x_i^k\searrow 0$ for each i, as $k\to\infty$, there exists an increasing sequence of indices $\{k_i\}_{i=1}^{\infty}$ such that $x_i^{k_i}\overset{0}{\to}0$ as $i\to\infty$. Let T be an abstract set and $\mathscr{T}$ a σ-algebra of subsets of T. Let $\mathscr{F}$ and $\mathscr{M}$ be vector lattices of type R^*, and suppose that there is an operator product $\langle x, y\rangle$ defined on $\mathscr{F}\times\mathscr{M}$ with values in a vector lattice $\mathscr{I}$ that satisfies a condition slightly weaker than R^*. Let $x=x(t)$ be a function defined on T with values in $\mathscr{F}$ and m a countably additive measure defined on $\mathscr{T}$ with values in $\mathscr{M}$. For some such functions, a Bochner type integral can be defined, in terms of the operator product $\langle x(t), m\rangle$, having its values in $\mathscr{I}$. For example, if $x=\sum c_i\chi_{A_i}$, where the sum is finite, the c_i are in $\mathscr{F}$, and χ denotes the characteristic function of a set, then one defines

$$\int_T x\,dm=\sum\langle c_i, m(A_i)\rangle.$$

This integral is then extended, just as in Bochner's theory. Most of the usual properties of an integral persist. Refined properties are obtained only under stringent assumptions on $\mathscr{F}$, $\mathscr{M}$, and $\mathscr{I}$.

Much of the material in this memoir has appeared in summary form elsewhere [see MR **15**, 721; **16**, 715; **17**, 177, 510, 989].

{It seems regrettable that this memoir was not written in a language more widely known than Romanian. There are many interesting notions in it, which are necessarily somewhat inaccessible to the majority of mathematicians.} *E. Hewitt* (Seattle, Wash.).

Bochner, S. Weak solutions of linear partial differential equations. J. Math. Pures Appl. (9) **35** (1956), 193–202.

Soit $T^1, \cdots, T^\mu$ des opérateurs différentiels donnés dans un ouvert S de R^n; soient $f^1, \cdots, f^\mu$ des fonctions localement sommables dans S, solutions faibles (i.e. au sens des distributions) de $T^1f^1+\cdots+T^\mu f^\mu=0$, dans $S_0=S-A$, A compact de S. Soit $\tilde{f}^i$ la fonction prolongée de f^i à S par 0 hors de S_0; si les fonctions f^i sont convenablement "nulles" sur A, alors $T^1\tilde{f}^1+\cdots+T^\mu\tilde{f}^\mu=0$ au sens des distributions sur S; la condition précise est

$$\int_{S^*\cap A_\varepsilon}|f^\lambda(x)|dx=o(\varepsilon^{N_\lambda})\ (\lambda=1, \cdots, \mu;\ \varepsilon\to 0),$$

où A_ε=ensemble des x à distance $\leqq\varepsilon$ de A, N_λ=ordre de T^λ. L'A. utilise ce résultat pour donner une condition permettant d'affirmer que si f est solution usuelle dans S_0 d'une équation aux dérivées partielles $Af=0$, á coefficients donnés dans S, alors f est solution faible de la même équations dans S. *J. L. Lions* (Nancy).

Bogolyubov, N. N.; and Parasyuk, O. S. On the analytic continuation of generalized functions. Dokl. Akad. Nauk SSSR (N.S.) **109** (1956), 717–719. (Russian)

On dira qu'une distribution T tempérée sur R [cf. L. Schwartz, Théorie des distributions, t. II, Hermann, Paris; 1951; MR **12**, 833] est prolongeable analytiquement dans le demi plan $y>0$ s'il existe une fonction $T(z)=T(x+iy)$ holomorphe dans $y>0$, telle que $x\to T(x+iy)$ definisse une distribution tempérée pour tout $y>0$ convergeant vers T lorsque $y\to 0$, et que $T(x+iy)$ soit majorée par un polynome en $|x|$, fixe, lorsque $y\geqq\delta>0$. La condition nécessaire et suffisante pour que T soit prolongeable analytiquement dans le demi plan $y>0$ est que la transformée de Fourier-Schwartz de T soit nulle sur le demi axe négatif.

Les A. énoncent ce résultat dans un cas particulier, i.e. avec des hypothèses supplémentaires sur T: ils démontrent probablement exactement ce qui leur est utile pour les applications à la mécanique quantique qu'ils ont en vue.

{Ce résultat n'est pas nouveau: c'est un cas particulier d'un résultat du rapporteur [cf. L. Schwartz, Medd. Lunds Univ. Mat. Sem. Tome Supplémentaire (1952), 196–206, prop. 8; MR **14**, 639; Lions, J. Analyse Math. **2** (1953), 369–380; MR **15**, 307].} *J. L. Lions.*

Landsberg, Max. Über das Spektrum symmetrisierbarer Endomorphismen in lokalkonvexen Räumen, insbesondere in Räumen vom Typ (ω). Math. Z. **66** (1956), 58–63.

The notion of a symmetrizable operator is generalized as follows: Let E be a complex locally convex Hausdorff topological linear space, E' its dual, and A a continuous linear mapping of E into itself. Then A is said to be symmetrizable, type 1, if there exists an antilinear mapping Θ of E' into E (i.e. $\Theta(u+v)=\Theta u+\Theta v$, $\Theta(\alpha u)=\bar{\alpha}\Theta u$) such that $\langle\Theta u, u\rangle\neq 0$ if $u \in E'$ and $u\neq 0$, and for each u and v in E', $\langle\Theta u, v\rangle=\overline{\langle\Theta v, u\rangle}$, $\langle A\Theta u, v\rangle=\overline{\langle A\Theta v, u\rangle}$. Here $\langle x, u\rangle$ is the value of u at x. The condition involving both

A and Θ is equivalent to $A\Theta=\Theta A'$, in view of the properties of Θ. Here A' is the dual of A. If E is a Hilbert space, A will be symmetrizable in this sense if there exists a self-adjoint operator H such that AH is self-adjoint and $(Hx, x)\neq 0$ if $x\neq 0$. The operator A is said to be symmetrizable, type 2, if A' is symmetrizable, type 1, on E'. It is understood throughout that E' is endowed with its weak topology. The space E itself is called symmetrizable, type 1 or 2, if the identity operator on E is thus symmetrizable.

By definition the spectrum $\sigma(A)$ consists of all complex λ such that $A-\lambda I$ fails to give a one-to-one mapping of E onto all of E. The set of eigenvalues of A is denoted by $\sigma_p(A)$, and $\sigma(A)-\sigma_p(A)=\sigma_k(A)$ is called the continuous spectrum for the purposes of this paper. Theorem 1: Suppose A is symmetrizable, type 1. Then each eigenvalue of A' is real and is also an eigenvalue of A. If, for each λ, $A'-\lambda I'$ is a homeomorphism of its domain onto its range, then $\sigma_k(A)$ is void. Theorem 2: If A is symmetrizable, type 1, and completely continuous in the sense of Leray [Acta Sci. Math. Szeged **12** (1950), Pars B, 177–186; MR **12**, 32], the points of $\sigma(A)$ are all real.

The space E is called of type (ω) if it is of infinite dimension and weakly complete relative to Cauchy filters. Every such space is symmetrizable, type 1, but not type 2, as may be shown by using results due to Köthe [Math. Ann. **120** (1949), 634–649; MR **10**, 610]. Theorem 3: For such a space and a symmetrizable (type 1) A, $\sigma_k(A)$ is void. Theorem 4: If in addition the dimension of E is $\aleph_0$, $\sigma(A)$ is nonvoid and so A has at least one eigenvalue. Theorem 5: If E is of type (ω), with $\dim E=\aleph_0$ and A symmetrizable, the following assertions are equivalent: (1) $\sigma(A)$ is real; (2) $\sigma(A)$ is at most countable; (3) $\sigma_p(A)=\sigma_p(A')$. This is proved by representing A with a column-finite matrix and using results due to Ulm [ibid. **114** (1937), 493–505]. *A. E. Taylor* (Los Angeles, Calif.).

Park, David. Operator methods in classical field theory. Indian J. Theoret. Phys. **3** (1955), 143–150.

It is shown how the solution of certain classical problems in the theory of fields — Poisson's integral and its Fourier transform, Lagrange's expansion, the Lienard-Wiechart potentials, Cauchy's problem etc. — can be treated in a simple and unified way by the use of operator methods, in which algebraic and analytic processes are suitably combined. The methods developed in the foregoing are then used to discuss an equation satisfied by retarded potentials only which is of the first order in the time derivative. *Author's summary.*

Vorob'ëv, Yu. V. The method of moments for non-self-adjoint linear operators and the acceleration of convergence of iterative processes. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 2(68), 161–167. (Russian)

Let A be a bounded linear operator on a Hilbert space H. The author extends to non-self-adjoint A ideas developed earlier for $A=A^*$ [Uspehi Mat. Nauk (N.S.) **10** (1955), no. 1(63), 89–96; MR **16**, 1124]. For $z_0\in H$, let $z_i=A^iz_0$, and let H_n be the subspace spanned by $z_0, \cdots, z_{n-1}$. Let H_z be the closure of the subspace spanned by the sequence $\{z_i\}$. Let B_n be the operator on H_n such that $z_k=B_n{}^kz_0$ $(k\leqq n-1)$ and $B_n{}^nz_0=$projection of z_n into H_n. The author proves the following Theorem: If $f(\lambda)$ is a holomorphic function for $\lambda<R>\|A\|$, then $f(B_n)$ converges strongly to $f(A)$ in H_z.

As an application the author considers finding the solution $x_*=(E-A)^{-1}f$ of the functional equation $x=Ax+f$, where $\|A\|=q<1$. Letting $f=z_0$, one can obtain B_n and solve the equation $x_n=B_nx_n+f$ for x_n. By the theorem, $|x_n-x_*\|\to 0$. Letting $P_n(\lambda)$ be the characteristic polynomial of B_n on H_n, the author finds the estimate
$$\|x_*-x_n\|\leqq(1-q)^{-1}\|P_n(A)f\|/|P_n(1)|.$$
(It is noted that one can compute $P_n(\lambda)$ and x_n without finding the eigenvalues or eigenvectors of B_n.) The method is interpreted as an acceleration of the classical iterative method of computing x_* as $\lim y_k$, where $y_{k+1}=Ay_k+f$. As an acceleration the author compares his method with that of Lyusternik [Trudy Mat. Inst. Steklov. **20** (1947), 49–64; MR **10**, 71], for which A must have a known eigenvalue of largest modulus.
G. E. Forsythe (Los Angeles, Calif.).

Civin, Paul; and Yood, Bertram. Ideals in multiplicative systems of continuous functions: a correction. Duke Math. J. **23** (1956), 631.

This note contains a correction to an earlier paper by the authors [same J. **23** (1956), 325–334; MR **17**, 1227]. The error occurred in Lemma 5.2 of that paper and the subsequent material of § 5 has been somewhat modified in this note. *R. S. Phillips* (Los Angeles, Calif.).

Iohvidov, I. S. Unitary operators in a space with an indefinite metric. Har'kov. Gos. Univ. Uč. Zap. **29**= Zap. Mat. Otd. Fiz.-Mat. Fak. i Har'kov. Mat. Obšč. (4) **21** (1949), 79–86. (Russian)

Consider the space H_k, consisting of all sequences $x=\{x_j\}_{j=1}^{\infty}$ of complex numbers such that $\sum_{j=1}^{\infty}|x_j|^2<\infty$. Topologize H_k with the usual l_2 metric, and introduce the usual pointwise linear operations. Define the inner product (x, y) in H_k by $(x, y)=-\sum_{j=1}^{k}x_j\bar{y}_j+\sum_{j=k+1}^{\infty}x_j\bar{y}_j$. This space was studied by L. S. Pontryagin [Izv. Akad. Nauk SSSR. Ser. Mat. **8** (1944), 243–280; MR **6**, 273]. Pontryagin proved, among other things, that every Hermitian operator on H_k admits a k-dimensional invariant subspace, in which the inner product (x, x) is non-positive, and such that all eigenvalues corresponding to this subspace have non-negative imaginary parts. The present paper gives an analogue of Pontryagin's theorem for unitary operators. A one-to-one map U of H_k onto itself is called unitary if $(x, y)=(Ux, Uy)$ for all $x, y\in H_k$. The author first shows that every unitary map is necessarily linear and continuous. Main theorem. Let U be a unitary operator on H_k. Then there is a k-dimensional subspace I of H_k that is invariant under U. The inner product (x, x) is non-positive on I, and all eigenvalues of U corresponding to elements of I have absolute value at least 1. The proof uses Pontryagin's theorem and the Cayley transform. *E. Hewitt* (Seattle, Wash.).

Sakai, Shôichirô. The absolute value of W^*-algebras of finite type. Tôhoku Math. J. (2) **8** (1956), 70–85.

This paper continues the program of approaching operator algebras on Hilbert space thru their linear and order properties [see Kadison, Ann. of Math. (3) **56** (1952), 494–503; MR **14**, 481]. The main result is that these properties together with a suitable inner product suffice to characterize Hilbert algebras with a unit. The formulation is such as to yield a multiplication on a special class of elements, but considerable development is required to make an adequate extension of this multiplication.

Specifically, let L be a complex Hilbert space with an

adjoint operation $*$ such that the real-linear space of self-adjoint elements a, $a=a^*$, are partially ordered in such a way that: (1) $(a, b)\geqq 0$ if $a\geqq 0$ and $b\geqq 0$; (2) there is defined an 'absolute value map' $a\to|a|$ on L, with $|a|\geqq 0$, having the properties (i) $\||a|\|=\|a\|$,

$$|(a, b)|\leqq(|a|, |b|)^{\frac{1}{2}}(|a^*|, |b^*|)^{\frac{1}{2}}$$

for arbitrary a and b, where $\|\cdot\|$ is the norm on L, (ii) for any a, b there is a c such that $(|a|, |b|)=(a, c)$ and $|c|=|b|$, (iii) $|a|\perp|b|$ implies $|a^*|\leqq|(a+b)^*|$ {the meaning of '$\perp$' in this connection is not clear to the reviewer}; (3) there is an element $I>0$ such that (i) $x\geqq 0$ and $x\perp I$ implies $x=0$; (ii) $|x|\leqq\alpha I$ implies $|x^*|\leqq\alpha I$ for α a positive number, (iii) $|x|\leqq\alpha I$ and $|y|\leqq\gamma I$ imply $|x+y|\leqq(\alpha+\gamma)I$ for α, γ positive numbers. Then there exists a linear isometry of L onto the Hilbert space of a Hilbert algebra with unit I' such that (1) $f(I)=I'\|I\|$, (2) $f(a)\geqq 0$ if and only if $a\geqq 0$, (3) $f(|a|)=|f(a)|$ for all a in L, where $|x|=(x^*x)^{\frac{1}{2}}$ for elements x of the Hilbert algebra. *I. E. Segal.*

See also: Jacobson, p. 373; Young, p. 384; Hlawka, p. 390; Arens and Eells, p. 406; Kyner, p. 408.

Banach Spaces, Banach Algebras

See: Trenogin, p. 383; Cotlar, p. 383; Kyner, p. 408.

Hilbert Space

See: Landsberg, p. 404; Vorob'ëv, p. 405; Sakai, p. 405; Kyner, p. 408.

TOPOLOGY

General Topology

Krasnosel'skiĭ, M. A. On special coverings of a finite-dimensional sphere. Dokl. Akad. Nauk SSSR (N.S.) **103** (1955), 961–964. (Russian)

Let A be a periodic mapping of the n-dimensional unit sphere on itself, satisfying two conditions: each of the mappings A, A^2, $\cdots$, A^{p-1} has no fixed point on S^n, and $A^p=I$, where I is the identity mapping. A closed set $F\subset S^n$ is called a set of the first kind if $AF=F$ and if in no connected component is there a pair x, A^ix, $1\leqq i\leqq p-1$. Denote by Π_n the set obtained from S^n by identifying the collection of points x, Ax, $\cdots$, $A^{p-1}x$. The author proves the following theorem. (I) Suppose the sets F_1, $\cdots$, F_r of the first kind form a covering of the sphere S^n. Then $r\geqq n+1$. (II) The category of the space Π_n is equal to $n+1$. *J. M. Danskin* (Princeton, N.J.).

Weiss, Edwin. Boundedness in topological rings. Pacific J. Math. **6** (1956), 149–158.

Most of the theorems in the paper concern a left bounded topological ring R: for any neighborhood U of 0 there exists a neighborhood V such that $VR\subset U$. Structure theorems are proved in the context of topological modification of purely algebraic notions. For instance, topological nilpotence, topological quasi-regularity, and topological regularity (in the sense of von Neumann) are introduced. *I. Kaplansky* (Princeton, N.J.).

Kodama, Yukihiro. Note on an absolute neighborhood extensor for metric spaces. J. Math. Soc. Japan **8** (1956), 206–215.

Kodama, Yukihiro. On sum theorems of ANR and a characteristic property of completely collectionwise normal spaces. Sci. Rep. Tokyo Kyoiku Daigaku. Sect. A. **5** (1956), 122–129.

A topological space X is an ANE (metric) if every continuous function from a closed subset B of a metric space Y into X can be extended continuously over a neighborhood of B; similarly for ANE (normal), etc. A space X has the weak topology with respect to a closed covering $\mathscr{A}$ if every subset S of X, which has a closed intersection with every member of some subfamily of $\mathscr{A}$ which covers S, is closed in X.

In the first paper, it is proved that if a space X has the weak topology with respect to a closed covering $\mathscr{A}$, and if every finite subcollection of $\mathscr{A}$ has an intersection which is an ANE (metric), then X is an ANE (metric). This generalizes the classical theorem of Aronszajn and Borsuk [Fund. Math. **18** (1932), 193–197], where $\mathscr{A}$ has only two elements. In the second paper, the same theorem is proved with ANE (metric) replaced by ANE (completely normal and fully normal) or, in case $\mathscr{A}$ is star-finite, by ANE (completely collectionwise normal).

Some interesting lemmas are obtained, such as the following from the second paper: If X is completely normal and fully normal, $A\subset X$ closed, and $\{A_\alpha\}$ a locally finite closed covering of A, then there exists a neighborhood S of A, and a locally finite closed covering $\{S_\alpha\}$ of S which is similar to $\{A_\alpha\}$, such that $S_\alpha\cap A=A_\alpha$ for all α. ($\{S_\alpha\}$ is similar to $\{A_\alpha\}$ if for any finite set of indices α_1, $\cdots$, α_n, $\bigcap_{i=1}^n A_{\alpha_i}=\emptyset$ if and only if $\bigcap_{i=1}^n S_{\alpha_i}=\emptyset$.)

E. Michael (Princeton, N.J.).

Arens, Richard F.; and Eells, James, Jr. On embedding uniform and topological spaces. Pacific J. Math. **6** (1956), 397–403.

Kuratowski [Fund. Math. **25** (1935), 534–545] gave a very simple method of embedding any bounded metric space M isometrically in a normed linear space, and Wojdysławski [Fund. Math. **32** (1939), 184–192] showed that the image of M under this embedding must be closed in its convex hull. A slight modification by Klee [Amer. Math. Monthly **58** (1951), 389–393; MR **13**, 147] similarly takes care of unbounded M. In the present paper, the authors obtain a new and more complicated isometric embedding of a metric space M in a normed linear space E which has the feature that the image of M is actually closed in E. An analogous embedding of a uniform space as closed subset of a locally convex linear space is also obtained. *E. Michael* (Princeton, N.J.).

Rhodes, F. A generalization of isometries to uniform spaces. Proc. Cambridge Philos. Soc. **52** (1956), 399–405.

It is known that a homeomorphism h of a uniform space preserves some basis for the uniform structure if and only if h generates a uniformly equicontinuous group

[Arens, Amer. J. Math. **68** (1946), 593–610; MR **8**, 479]. This is here generalized by the theorem that a uniformly equicontinuous semigroup (of homeomorphisms, as above) consists wholly of contractions relative to some basis for the uniform structure. This, in turn, rests upon a generalization of Freudenthal and Hurewicz' theorem [Fund. Math. **26** (1936), 120–122]. Concerning any compact transformation group of a connected Hausdorff space, it is proved that it is transitive if the orbit of some point x covers a neighborhood of x. *R. Arens.*

Ginsburg, Seymour. Sets which are not homeomorphic by m-decomposition. Ann. of Math. (2) **64** (1956), 447–449.

The space E is assumed to be metric separable and of power $2^{\aleph_0}$. Two sets X and Y ($X\subset E$, $Y\subset E$) are said to be homeomorphic by m-decompositions if for some index set R of power $m<2^{\aleph_0}$ there exists a decomposition $X=\bigcup_{r\epsilon R} X_r$, $Y=\bigcup_{r\epsilon R} Y_r$, $X_{r_1}\cap X_{r_2}=Y_{r_1}\cap Y_{r_2}$ is empty for $r_1\neq r_2$ so that X_r and Y_r are homeomorphic.

Assume that m is not cofinal with $2^{\aleph_0}$; the author shows that there exists a family F of $2^{2^{\aleph_0}}$ subsets of E each of power $2^{\aleph_0}$ so that no two elements of F are homeomorphic by m-decompositions. [Cf. previous results on equivalence by m-decompositions by the reviewer, Ann. of Math. (2) **44** (1943), 643–646; MR **5**, 173; and Sierpiński, ibid. **48** (1947), 641–642; MR **9**, 17.]

Several other results are obtained by the author.

P. Erdös (Birmingham).

Bing, R. H. A simple closed curve that pierces no disk. J. Math. Pures Appl. (9) **35** (1956), 337–343.

The author constructs in 3-space S a simple closed curve J such that any simple closed curve in $S-J$ that bounds a disc whose intersection with J is finite must also bound a disc that does not intersect J at all. In discussing this example he notes that there exist in S disjoint simple closed curves J_1 and J_2 such that J_1 bounds a disc disjoint to J_2 but J_2 bounds no disc disjoint to J_1. *R. H. Fox* (Princeton, N.J.).

Anderson, R. D. Some remarks on totally disconnected sections of monotone open mappings. Bull. Acad. Polon. Sci. Cl. III. **4** (1956), 329–330.

Let f be a monotone open mapping of a compact metric space X onto a compact metric space Y such that $f^{-1}[y]$ consists of more than one point for each y in Y. A closed totally disconnected subset K of X is called a section if $f[K]=Y$. It was proved by the reviewer that if $\dim(Y)<\infty$ then a section always exists; the author shows that: (a) If X is either one dimensional or a subset of a 2-manifold a section always exists. (b) For $n\geqq 2$ there is an n-dimensional X, a function f and space Y of the prescribed sort with no section. (c) For $n>3$ there is an n-manifold X, a function f and a space Y with no section. In particular, local connectivity of X does not imply the existence of a section. *J. L. Kelley* (Berkeley, Calif.).

Weier, Joseph. On plane vector fields. Math. Japon. **3** (1955), 163–172.

Let P denote the closure of a Jordan region in the plane, and suppose f and g are maps of P into the one-dimensional sphere Q. If F, G denote the homotopy classes of maps of P into Q containing f, g respectively, then a sufficient condition is given that for every $f_1\epsilon F$ and $g_1\epsilon G$ there exist $p\epsilon P$ with $f_1(p)=g_1(p)$. *E. E. Floyd.*

Reeb, G. Sur la théorie générale des systèmes dynamiques. Ann. Inst. Fourier, Grenoble **6** (1955–1956), 89–115.

This paper develops a major portion of the concepts of topological dynamics without use of transformation groups. Equivalence relations play an essential rôle. On the basis of an equivalence relation alone, the notions of trajectory, invariant set, minimal set, indecomposable set, central motions and proper and improper trajectories (these are related to the classic concept of Poisson stability), can be defined and elementary properties can be derived. If the relation is assumed to be open, additional theorems can be derived. As an example — if the space under consideration is compact and second countable, then any indecomposable subset contains an everywhere dense trajectory. The introduction of a local structure permits the study of Poisson stability and its relationship to improper trajectories and other properties, the definition of recurrence (in the sense of G. D. Birkhoff) and its relation with minimality, as well as the notions of wandering trajectory and incompressibility. The structures defined here and those associated with topological dynamics are not identical classes, but there is a large common ground. Examples are given which illustrate some of the possibilities and which point up the place of transformation groups in the classic dynamic systems.

G. A. Hedlund (New Haven, Conn.).

Budak, B. M. The concept of motion in a generalized dynamical system. Moskov. Gos. Univ. Uč. Zap. **155**, Mat. **5** (1952), 174–194. (Russian)

A dispersive dynamical system is a family $f(p, t)$ $(-\infty<t<\infty)$ of one-many mappings of a complete separable metric space into itself, satisfying six conditions, for which see the review of an earlier paper by the author [Vestnik Moskov. Univ. **1947**, no. 8, 135–137; MR **10**, 309], where the definition of a motion $\varphi_p(t)$ through p is also given. An ordinary dynamical system $dx/dt=X(x)$ in n-space, X merely continuous, is shown to be an example of a dispersive system, where $f(p, t)$ is the set of points reached at time t on some solution curve starting from p. A point p is called a point of continuity of the system if $\alpha(f(p_n, t_n), f(p, t))\to 0$ whenever $p_n\to p$ and $t_n\to t$, where $\alpha(A, B)$ denotes the infimum of the positive numbers ϵ such that each of the sets A and B is contained in the ϵ-neighborhood of the other. The results are largely concerned with the existence and convergence of motions under various hypotheses. Typical are: Theorem 1. If $q\epsilon f(p, t)$ there exists a segment of a motion joining p to q. Theorem 5. If p and the points of a motion $\varphi_p(t)$ corresponding to some set of t-values dense on the real line J are points of continuity of the system, then for any sequence p_n converging to p there exist motions $\varphi_{p_n}(t)$ issuing from these points which converge uniformly to $\varphi_p(t)$ on any finite t-interval. Other results concern the compactness or connectedness of the funnel (orbit) $f(p, J)$ with vertex p, α- or ω-limiting sets, and quasi-invariant sets F, of two types according as $F\subset f(F, t)$ or $F\supset f(F, t)$ for all t. *J. C. Oxtoby* (Bryn Mawr, Pa.).

Trucco, Ernesto. On the information content of graphs: compound symbols; different states for each point. Bull. Math. Biophys. **18** (1956), 237–253.

The author generalizes the "information content" of a graph defined by N. Rashevsky [same Bull. **17** (1955), 229–235; MR **17**, 280]. *W. T. Tutte* (Toronto, Ont.).

Dirac, G. A. Map colour theorems related to the Heawood colour formula. J. London Math. Soc. 31 (1956), 460–471.

It is known that for $h>1$ a map on a surface of connectivity h has chromatic number at most

$$H_h=[3\tfrac{1}{2}+\tfrac{1}{2}\sqrt{(24h-23)}].$$

It is conjectured that any map of chromatic number k either contains k mutually adjacent countries or can be transformed into a map with this property by deleting one or more frontier lines. The author shows that the conjecture is true for (H_h-1)-chromatic maps if $h\geq 6$, and for (H_h-2)-chromatic maps if $h\geq 13$ and $h\neq 14$, 15, 18, 19, 22, 26, 31. In the case $h\geq 6$, $h\neq 7$, 9, 10 he shows that any (H_h-1)-chromatic map includes a set of countries related by their common frontier lines in one of two specified ways. *W. T. Tutte* (Toronto, Ont.).

Kotzig, Anton. The significance of the skeleton of a graph for the construction of composition bases of some subgraphs. Mat.-Fyz. Časopis. Slovensk. Akad. Vied 6 (1956), 68–77. (Slovakian. Russian summary)

The terminology of the paper is that of D. König [Theorie der endlichen und unendlichen Graphen, Akademische Verlagsgesellschaft, Leipzig, 1936]. Let G be a finite connected graph. First it is shown essentially that there may exist composition bases of circles of G that are not fundamental systems of circles of any skeleton of G and composition bases of circles of G that are fundamental systems of circles of more than one skeleton of G.

A "cut" of G is a subgraph R having the following properties: if we cancel in G all [resp. all except an arbitrary one] edges $\in R$, we get a non-connected [resp. a connected] graph. Let S be a skeleton of G. To every edge $h\in S$ there exists exactly one cut R_h containing h whose remaining edges do not belong to S. This defines a mapping of the set of edges of S to a certain system of cuts of G which the author calls "fundamental system of cuts belonging to the skeleton S". A fundamental system of cuts belonging to any skeleton of G is a composition basis of all cuts of G. Further results concerning composition bases of all cuts of G are given. *Št. Schwarz.*

Erdös, Paul. Some theorems on graphs. Riveon Lematematika 9 (1955), 13–17. (Hebrew. English summary)

A graph G_n of n vertices and $[n^2/4]+l$ edges, $l\leq 3$, $l<n/2$, contains t triangles, $t\geq l[n/2]$ [$t\geq 1$ by Turán, Mat. Fiz. Lapok 48 (1941), 436–452; Colloq. Math. 3 (1954), 19–30; MR 8, 284; 15, 976]. For even n, $l=1$, this was proved by Rademacher (orally to Erdös). All G_n with $l=1$, $t=[n/2]$ are isomorphic. *T. S. Motzkin* (Los Angeles, Calif.).

Gilbert, E. N. Enumeration of labelled graphs. Canad. J. Math. 8 (1956), 405–411.

A formula is obtained for the number of connected graphs having V labelled vertices and λ unlabelled edges. Loops, and circuits of two edges, may or may not be allowed, and the graphs may be either oriented or unoriented; the formula covers all eight cases.

The author points out that graphs with labelled edges and unlabelled vertices can be dealt with in the same way. He obtains a formula for the number of even graphs with L labelled edges and λ unlabelled vertices.

W. T. Tutte (Toronto, Ont.).

Tutte, W. T. A theorem on planar graphs. Trans. Amer. Math. Soc. 82 (1956), 99–116.

The author proves that if G is any planar graph, E is an edge of G which is not an isthmus, and E' is an edge distinct from E of a terminal circuit of E, then G contains a circuit J having the following properties: (i) E and E' are both in J. (ii) A bridge over J has at most three nodes in common with J, and if the bridge contains an edge belonging to a terminal circuit of E then it has exactly two nodes in common with J. It is then deduced that if G is planar and 4-connected and has more than one edge then G is Hamiltonian. Hence we have a generalization of Hassler Whitney's theorem on Hamiltonian circuits of plane triangulations [Ann. of Math. (2) 32 (1931), 378–390]. *G. A. Dirac* (Vienna).

★**Kyner, Walter T. A fixed point theorem.** Contributions to the theory of nonlinear oscillations, vol. 3, pp. 197–205. Annals of Mathematics Studies, no. 36. Princeton University Press, Princeton, N. J., 1956. \$4.00.

Consider the Banach space of real m-dimensional, continuously differentiable, multiply periodic vector functions $f=(f_1, f_2, \cdots, f_m)$ of the real s-dimensional vector $\theta=(\theta_1, \theta_2, \cdots, \theta_s)$ with norm

$$\|f\|=\max_{i,j,k}\left(\max_{\theta}|f_i(\theta)|, \max_{\theta}\left|\frac{\partial f_j}{\partial\theta_k}\right|\right).$$

Consider the transformation

$$[T_\gamma f](\bar\theta)=N(\theta)f(\theta)+W(f(\theta), \theta, \gamma),$$

where: $\bar\theta=V_{f,\gamma}(\theta)=\theta+\pi+U(f(\theta), \theta, \gamma)$; π is a constant vector; $U(y, \theta, \gamma)$ and $W(y, \theta, \gamma)$ have second partial derivatives with respect to y and θ which are continuous in y, θ and γ, are periodic in θ and satisfy

$$U(0, \theta, 0)\equiv W(0, \theta, 0)\equiv\frac{\partial W}{\partial y}(0, \theta, 0)\equiv 0;$$

and $N(\theta)$ is an m by m matrix whose elements are C^2 functions of θ. Define the linear transformation

$$[L_{f\gamma}g](\theta)=g(V_{f,\gamma}(\theta))-N(\theta)g(\theta)$$

and suppose $L_{f\gamma}$ has a uniformly bounded inverse for all $\|f\|\leq r_0$ and $|\gamma|\leq\gamma_0$. Under these hypotheses the author proves that there exists a $\gamma_1>0$ such that if $|\gamma|\leq\gamma_1$, then T_γ has a unique fixed point f_γ which is continuous in γ and $f_\gamma\to 0$ as $\gamma\to 0$. Moreover $\partial f_\gamma/\partial\theta_k$ satisfies a uniform Lipschitz condition. The result generalizes some fixed point theorems used by G. Hufford [Thesis, Princeton, 1953] and M. Marcus [Thesis, Univ. of California, 1954] in perturbation analysis of ordinary differential equations.

C. E. Langenhop (Ames, Ia.).

Ghéorghiev, Ghéorghi Iv. Une condition suffisante pour l'existance du point fixe pour certains automorphismes du plan. Univ. d'Etat Varna "Kiril Slavianobâlgarski" Fac. Tech. Méc. Annuaire 3 (1947–1948), 47–51 (1949). (Bulgarian. French summary)

The author proves that every homeomorphism of the plane onto itself which maps some simple closed curve onto itself has a fixed point. His proof consists of showing that the theorem is an easy consequence of the Jordan curve theorem and the Brouwer fixed-point theorem.

Eldon Dyer (Princeton, N.J.).

See also: Uspenskiĭ, p. 369; Padmavally, p. 373; Klement'ev, p. 384; Young, p. 384; Tsuji, p. 386; Marcus, p. 394; Słowikowski and Zawadowski, p. 403; Klee, p. 404; Landsberg, p. 404; Deards, p. 442.

Algebraic Topology

Postnikov, M. M. Homology theory of smooth manifolds and its generalizations. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 1(67), 115–166. (Russian)

This paper gives a detailed and very readable account of Fáry's version [C. R. Acad. Sci. Paris **237** (1953), 552–554; MR **15**, 147] of Leray's uniqueness theorem for cohomology with compact carriers, based on the concept of "couverture". All definitions are carefully introduced, beginning with graded differential module and algebra, the homology of such a structure, algebra with supports in a space (Leray's "complex"), algebra with compact supports, "couverture", Fary's theorems. Along with this is developed, and used as motivation, the theory of differential forms in a smooth manifold so that de Rham's theorems are proved. The "Alexander algebra" [cf. E. H. Spanier, Ann. of Math. (2) **49** (1948), 407–427; MR **9**, 523] is studied, as well as a subalgebra considered by Fary; it is noted that the Aleksandrov-Čech Theory can be given a form that fits into the present frame. (The definition of the Alexander algebra contains several misprints.) *H. Samelson* (Ann Arbor, Mich.).

Al'ber, S. I. Homologies of the spinor group. Dokl. Akad. Nauk SSSR (N.S.) **104** (1955), 341–344. (Russian)

The author determines the integral homology of the spinor groups and gives a geometrical construction of all the cycles [as far as Betti numbers and torsion coefficients are concerned, the results are contained implicitly in those of A. Borel, Proc. Nat. Acad. Sci. U.S.A. **39** (1953), 1142–1146; MR **15**, 505]. A basis for the Betti group is constructed, which consists of homeomorphs of manifolds which are covered by products of spheres. The individual spheres are mapped by the standard characteristic map into the orthogonal group, lifted into the spinor group; the spheres are multipled by Pontryagin multiplication. The torsion cycles are constructed in a similar fashion. The procedure is inductive, and uses the inclusion $\mathrm{Spin}\,(n-1)\subset\mathrm{Spin}\,(n)$; it is proved that for $n\neq 2^s+1$ this gives an isomorphism mod 2; for $n=2^s+1$, the kernel is exhibited. *H. Samelson.*

Suzuki, Haruo. On the Eilenberg-MacLane invariants of loop spaces. J. Math. Soc. Japan **8** (1956), 93–101.

Let X be a simply-connected topological space, $x_0\in X$ a base point, and Ω the space of loops on X based at x_0. Suppose further that $\pi_i(X)=0$ for $i<p$ and $p<i<q$, where $p<q$, so that Eilenberg-Maclane invariants

$$k(X)\in H^{q+1}(\pi_p,\,p;\,\pi_q),\quad k(\Omega)\in H^q(\pi_p,\,p-1;\,\pi_q)$$

are defined, where $\pi_p=\pi_p(X)$, $\pi_q=\pi_q(X)$. Now there is a suspension homomorphism

$$S{:}H^t(G,\,p;\,A)\to H^{t-1}(G,\,p-1;\,A)$$

and the author proves the well-known but perhaps hitherto unpublished result that $k(\Omega)=\pm Sk(X)$. The sign will, of course, depend on the conventions adopted in the course of making the relevant definitions.

A similar result is proved for the Postnikov invariants (the above is, of course, a special case), using J. H. C. Whitehead's formulation [Proc. London Math. Soc. (3) **3** (1953), 385–416; MR **15**, 734]. Conclusions are drawn about relations between the q-type of Ω and the $(q+1)$-type of X, but the statement on p. 93 that, if X is a CW-complex and q sufficiently small, then the homotopy type of Ω determines that of X, is false (there may be a linguistic difficulty here).

The author's notation involves him in such unfortunate expressions as $\pi_p=\pi_{p-1}$, $\pi_q=\pi_{q-1}$; however, it failed to conceal the misprint on p. 94, l. 7, where 'q-cells' should read 'n-cells'. *P. J. Hilton* (Manchester).

★ Séminaire Henri Cartan de l'Ecole Normale Supérieure, 1949/1950. Espaces fibrés et homotopie. 18 exposés par Blanchard, A.; Borel, A.; Cartan, H.; Serre, J. P.; and Wen Tsün, Wu. 2ème éd. Secrétariat mathématique, 11 rue Pierre Curie, Paris, 1956. iii+149 pp.

This set of seminar notes can be considered as a text on the subjects of homotopy theory and the theory of fibre spaces. A good idea of the material covered can be obtained from the titles of the 18 exposés: 1.– J. P. Serre: Extension des applications. Homotopie; 2.– J. P. Serre: Groupes d'homotopie; 3.– H. Cartan: Problèmes d'homotopie et de prolongement: théorie des obstructions; 4.– H. Cartan: Applications d'espaces localement compacts dans des polyèdres: dimension, problèmes d'homotopie et de prolongement; 5.– A. Blanchard: Exemples d'espaces fibrés; 6.– H. Cartan: Généralités sur les espaces fibrés, I; 7.– H. Cartan: Généralités sur les espaces fibrés, II; 8.– H. Cartan: Généralités sur les espaces fibrés, III; 8bis.– H. Cartan: Généralités sur les espaces fibrés (appendice); 9.– J. P. Serre: Groupes d'homotopie relatifs. Application aux espaces fibrés; 10.– J. P. Serre: Homotopie des espaces fibrés. Applications; 12.– A. Borel: Groupes d'homotopie des groupes de Lie, I; 13.– A. Borel: Groupes d'homotopie des groupes de Lie, II; 14.–H. Cartan: Carrés de Steenrod, I; 15.–H . Cartan: Carrés de Steenrod, II; 17.– Wu Wen-Tsün: Les classes caractéristiques d'un espace fibré I: Cohomologie des grassmanniennes; 18.– Wu Wen-Tsün: Les classes caractéristiques d'un espace fibré II; 19.– H. Cartan: Cohomologie réelle d'un espace fibré principal différentiable. I: Notions d'algèbre différentielle, algèbre de Weil d'un groupe de Lie; 20.– H. Cartan: Cohomologie réelle d'un espace fibré principal différentiable. II: Transgression dans un groupe de Lie et dans un espace fibré principal; recherche de la cohomologie de l'espace de base. Exposés 11 and 16 have not been mimeographed. (No reason is given.)

The first four exposés give an excellent introduction to the fundamentals of homotopy theory. Although all the material in these exposés is well known to workers in this field and is available in published papers, so far it has not been collected together in a text (the text by P. J. Hilton [An introduction to homotopy theory, Cambridge, 1953; MR **15**, 52] does not contain anything about the theory of obstructions).

Exposé 5 is intended to give the reader some motivation and background material for the sections on fibre spaces which follow.

The next three exposés by H. Cartan give rigorous definitions of the basic concepts of the theory of fibre spaces and establish the fundamental properties of fibre spaces. Since the method of presentation used here is not as standard and well-known as is the case with most of the other exposés of this seminar, it is perhaps wothwhile

to review these three exposés at somewhat greater length.

The most general definition given of the term "fibre space" is the following: A fibre space is a triple (E, B, p) such that E and B are topological spaces, p is a continuous, open map of E onto B, and the fibres (the subspaces $p^{-1}(b)$ for $b \in B$) are all homeomorphic. The notions of homomorphism and isomorphism of fibre spaces are then defined in the obvious way. A fibre space is called "trivial" if it is isomorphic to a product space (in the large); it is called "locally trivial" if each point of the base space B has a neighborhood such that the induced fibre space over this neighborhood is isomorphic to a product space. In none of his definitions does the author require that a fibre space should be locally trivial. Apparently one of his objectives is to eliminate the hypothesis of local triviality wherever possible.

Next, the definition of a "principal fibre space" is given. A principal fibre space is a topological space E with a topological group G operating (on the right) in such a way that the following axiom is satisfied: The graph C of the equivalence relation defined by G is a closed subset of $E \times E$; for each point $(x, y) \in C$ there exists a unique element $s \in G$ such that $x \cdot s = y$; finally, if we denote by u the map of C into G thus defined, the function u is required to be continuous. Of course it is necessary to verify that if $B = E/G$ denotes the space of orbits, and $p: E \to B$ is the natural projection, then (E, B, p) is a fibre space in the previously defined sense. A "fibre space with a structural group" is now defined to be a certain fibre space associated with a principal fibre space, as follows. Let H be a principal fibre space with group G, base B, and projection $q: H \to B$. Let F be a topological space on which G operates on the left; F is to be the fibre. Let G operate (on the right) on the product space $H \times F$ according to the following rule: $(h, f) \cdot g = (h \cdot g, g^{-1} \cdot f)$ for $h \in H$, $f \in F$, and $g \in G$. This defines $H \times F$ as a principal fibre space with group G. Let E denote the base of this principal fibre space. The projection $q: H \to B$ induces a map $p: E \to B$, and (E, B, p) is the desired fibre space with fibre F and structural group G. Note that all these definitions are formulated in an invariant fashion; it is not necessary to choose an open covering of the base space, coordinate functions, etc. If convenient, one may require that a principal fibre space or a fibre space with structural group be locally trivial. Various theorems assert that a principal fibre space or a fibre space with a structural group is locally trivial provided the base space and structural group satisfy certain hypotheses. The "fibre bundles" considered in the book by N. E. Steenrod [The topology of fibre bundles, Princeton, 1951; MR 12, 522] are "locally trivial fibre spaces with a structural group" in the terminology of the present author.

With these definitions established, Cartan considers some of the standard topics in the theory of fibre spaces, e.g. cross sections, extension and restriction of the structural group, the covering homotopy theorem, and classifying spaces. Among the theorems proved is the following: Any locally trivial fibre space (in the most general sense mentioned above) with the fibre locally compact can be considered as a fibre space with a structural group G (take G to be the group of all homeomorphisms of the fibre with an appropriate topology). The appendix to exposé 8 considers fibre spaces for which the structural group does not have any topology assigned to it, similar to those considered by Ehresmann and Feldbau [C. R. Acad. Sci. Paris **212** (1941), 945–948; MR 3, 58].

Exposés 9 and 10 are concerned with the definition of the relative homotopy groups, the homotopy sequence of a pair, and the exact homotopy sequence of a fibre space. These concepts are applied to obtain information about the homotopy groups of covering spaces, homogeneous spaces, Stiefel manifolds, and spheres. The treatment of these topics is fairly standard.

Exposés 12 and 13 give a rather complete survey of the facts known about the homotopy groups of Lie groups in 1949. Included are proofs of the fact that the fundamental group of a compact, simple, non-abelian Lie group is finite and that the second homotopy group is trivial. The structure of the first five homotopy groups of the classical groups is given (without proof).

Exposés 14 and 15 give the definitions and principal properties of the Steenrod squares, whether in a simplicial polyhedron or in an arbitrary space with singular, Čech, or Alexander-Spanier cohomology. Most of the exposition is based on Steenrod's fundamental paper [Ann. of Math. (2) **48** (1947), 290–320; MR **9**, 154]. Also included are complete proofs of results announced by Cartan in a note [C. R. Acad. Sci. Paris **230** (1950), 425–427; MR **12**, 42].

Exposé 17 is concerned with the cohomology of Grassmann varieties. The determination of the modulo 2 homology groups of the Grassmann varieties by Ehresmann [J. Math. Pures Appl. (9) **16** (1937), 69–100] by means of a cellular subdivision and the determination of the modulo 2 cohomology ring of a Grassmannian by Chern [Ann. of Math. (2) **49** (1948), 362–372; MR **9**, 456] are given without proof. Also included is a proof of the Whitney duality theorem of sphere bundles. In exposé 18 the definition of Stiefel-Whitney classes of a sphere bundle as obstructions to cross sections is given. A proof of the fact that the Stiefel-Whitney classes reduced modulo 2 may also be obtained as inverse images of generators of the cohomology ring of the Grassmann variety is briefly sketched.

The material of exposés 19 and 20 has been published by Cartan elsewhere [Colloque de topologie (espaces fibrés), Bruxelles, 1950, Thone, Liège, 1951, pp. 15–27, 57–71; MR **13**, 107]. The contents of these last two exposés is more of the nature of an original research paper, in strong contrast to all the preceding exposés.

Throughout these seminar notes the standards of exposition are high — much higher in fact than one usually finds in mimeographed sets of notes. They will undoubtedly be valuable as a reference work, and as an aid to the student or mature mathematician who wants to study the theory of fibre spaces and homotopy theory. It is unfortunate that the first edition was not available for sale to the public seven years ago. *W. S. Massey.*

Karpelevič, F. I.; and Oniščik, A. L. Algebra of homologies of a space of paths. Dokl. Akad. Nauk SSSR (N.S.) **106** (1956), 967–969. (Russian)

Let X be a simply connected space, and let Ω be its space of loops, based at some point of X. Theorem: If the cohomology ring $H^*(X)$ (with coefficients in a field of characteristic 0) is an exterior algebra with generators $x_1, \cdots, x_n$, then $H^*(\Omega)$ is a polynomial ring with generators $\xi_1, \cdots, \xi_n$, $\deg \xi_i = \deg x_i - 1$. The proof consists of computations with the spectral sequence of the usual space of paths in X, utilizing the fact that $H^*(\Omega)$ is tensor product of a polynomial ring and an exterior algebra, since Ω is an H-space. *H. Samelson.*

See also: Bing, p. 407.

GEOMETRY

Geometries, Euclidean and other

Rajagopal, A. K. A direct derivation of the equation of the director circle of an ellipse. Math. Mag. 30 (1957), 158–159.

Labra, M. Homological properties of pedal triangles. Rev. Ci., Lima 57 (1955), 71–87. (Spanish)

Given the triangle $(T)=ABC$ and a point P, the triangle $(T')=A'B'C'$ formed by the points $A'=(AP, BC)$, $B'+(BP, CA)$, $C'+(CP, AB)$ is the pedal, or cevian triangle of P for (T). The point P has a cevian triangle (T'') for (T'), a cevian triangle (T''') for (T''), etc. The point P is thus, by construction, the center of homology of any two of the triangles considered. The author proves analytically that the axis of homology of (T), (T') [which line he calls the Desargues line of P for (T) and which is commonly referred to as the trilinear polar of P for (T)] is also the axis of homology of any two of the triangles considered.

The author also proves analytically that the trilinear polars of the orthocenter, the circumcenter, and the Lemoine point for (T) are concurrent, and the same holds for the trilinear polars of the incenter, the orthocenter, the Nagel point, and the Gergonne point.

N. A. Court (Norman, Okla.).

Goormaghtigh, R. Triangle inscrit à un triangle. Mathesis 65 (1956), 428–429.

Prompted by the relations which J. Satterly [Math. Gaz. 40 (1956), 109–113] gave of elements connected with three points D, E, F dividing the sides BC, CA, AB of a triangle ABC in the same ratio, the author quotes analogous relations, when those three ratios are unequal. He concludes by proving the following property, not mentioned by Satterly: If the points D, E, F divide the sides of a triangle in the same ratio, the triangles ABC, DEF, and the triangle formed by the lines AD, BE, CF, have the same Brocard angle.

N. A. Court.

Thébault, V. Sur l'inversion triangulaire. Mathesis 65 (1956), 417–418.

Let F, F' be a pair of isogonal conjugate points for a triangle ABC; $A'B'C'$ — the pedal triangle of F for ABC (i. e., the triangle formed by the orthogonal projections of F upon the sides of ABC); $A''B''C''$, $A_1B_1C_1$ — the pedal triangles of F, F' for the triangles $A'B'C'$, ABC, respectively.

The following two properties are obtained. 1. The pairs of triangles $FA''A'$, $F'AA_1$; $FB''B'$, $F'BB_1$; $FC''C'$, $F'CC_1$, are similar. 2. $F'A\cdot FA''=F'B\cdot FB''=F'C\cdot FC''=\beta^2$, where β is the non-focal semi-axis of the conic inscribed in T and having F, F' for its foci.

N. A. Court (Norman, Okla.).

van Yzeren, J. Isogonal relations for complete quadrilaterals and for the Malfatti configuration. Simon Stevin 31 (1956), 19–26. (Dutch)

If B_4 is the isogonal point of A_4 with respect to the triangle $A_1A_2A_3$ and B_1 the mirror point of B_4 in A_2A_3, then B_1 is the isogonal point of A_1 with respect to the triangle $A_2A_3A_4$. Application to the generalized Malfatti configuration gives a simplified construction for the latter.

O. Bottema (Delft).

Toscano, Letterio. Confronto degli angoli di un triangolo con quelli del triangolo delle sue mediane. Archimede 8 (1956), 278–279.

Thébault, V. Sphères associées à un tétraèdre. Mathesis 65 (1956), 426–428.

Rodeja F., E. G.-. On pseudo-isosceles triangles. Gac. Mat., Madrid (1) 8 (1956), 65–68. (Spanish)

Fissato un lato di un triangolo, si determinano gli intervalli entro i quali possono variate gli altri elementi del triangolo affinchè questo possa essere pseudo-isoscele, ossia siano uguali le bisettrici esterne relative ai vertici del lato fissato.

C. Longo (Parma).

Thébault, Victor. Recreational geometry: The triangle. Scripta Math. 22 (1956), 14–30.

A selection of topics from the modern geometry of the triangle, beginning with Neuberg and others in the second half of the nineteenth century.

Gaddum, J. W. Distance sums on a sphere and angle sums in a simplex. Amer. Math. Monthly 63 (1956), 91–96.

The sum of spherical distances of k points on the unit sphere in n-space is $\leq[k^2/4]\pi$, for global points $\geq(k-1)\pi$. The sum of dihedral angles of a spherical simplex is $\leq\binom{n-1}{2}\pi+2(n-1)\pi a$ and $\geq[\frac{1}{4}(n-1)^2]\pi+[\frac{1}{2}n^2]a$, where a is the relative area (area divided by that of sphere); for a hyperplanar simplex set $a=0$.

T. S. Motzkin.

Stefánsson, Sigurkarl. A theorem on the diameters of a parabola, with applications. Nordisk Mat. Tidskr. 4 (1956), 189–194, 229. (Danish. English summary)

Lombardo-Radice, Lucio. Sul problema dei k-archi completi in $S_{2,q}$. ($q=p^t$, p primo dispari.) Boll. Un. Mat. Ital. (3) 11 (1956), 178–181.

In a finite plane of order n a k-arc is a set of k points no three on a line. A k-arc is said to be complete if it is not part of a $k+1$-arc. If n is odd the maximum possible k is $n+1$ and for a Desarguesian plane where $n=p^r$ an $n+1$-arc has been shown by Segre [Canad. J. Math. 7 (1955), 414–416; MR 17, 72] to be a non-degenerate conic. For Desarguesian planes of prime order q a method is given for constructing $\frac{1}{2}(q+5)$-arcs which are not part of a conic. For $q=3 \pmod 4$ and $q\geq 7$ a complete $\frac{1}{2}(q+5)$-arc is constructed, using quadratic residues of q.

Marshall Hall, Jr. (Columbus, Ohio).

Pickert, Günter. Eine nichtdesarguessche Ebene mit einem Körper als Koordinatenbereich. Publ. Math. Debrecen 4 (1956), 157–160.

In an arbitrary affine plane we may introduce coordinates from a system S so that a point is represented by (x, y) for $x, y \in S$. This may be done so that all points (a, a) lie on a line, $y=x$, all points $(a, 0)$ line on a line, the x-axis, and all points $(0, a)$ lie on a line, the y-axis. All points on a line parallel to the y-axis have the same x-coordinate, all points on a line parallel to the x-axis have the same y-coordinate. In S an addition and multiplication are defined, saying that $y=x+b$ if (x, y) is on a line parallel to $y=x$ through the point $(0, b)$ and saying

that $y=xm$ if (x, y) is on the line joining $(0, 0)$ and $(1, m)$. If the plane is Desarguesian, then 1) with respect to the defined addition and multiplication S is a division ring; and 2) every line not parallel to the y-axis consists of the points (x, y) satisfying a linear equation $y=xm+b$.

The author asks whether 1) implies 2). He shows that this is not the case and gives an example. This example is of the Moulton type in which lines of a Desarguesian plane are replaced by broken lines. *Marshall Hall, Jr.*

Curzio, Mario. Una osservazione sui piani grafici h–l transitivi. Boll. Un. Mat. Ital. (3) **11** (1956), 238–241.

The author shows that if a plane has all possible homologies with center on a line h and axis l, i.e. h–l transitive, and if further it has all possible homologies with center a point U not on h, and axis l, then the plane is indeed Desarguesian. This he proves by showing that the plane has every homology with axis l, and it was previously known that in this case the plane is Desarguesian. *Marshall Hall, Jr.* (Columbus, Ohio).

Mayer, O. Familles R de surfaces transversales dans les congruences de droites. Acad. R. P. Romîne. Fil. Iaşi. Stud. Cerc. Şti. **6** (1955), no. 1–2, 69–89. (Romanian. Russian and French summaries)

The author considers congruences of straight lines in the projective n-space whose two focal surfaces are distinct. These can be imbedded in a family R of transversal surfaces (Σ) having the following properties. 1) Every surface Σ intersects the rays of the congruence in one point each. 2) The correspondence thus determined between any two rays of the congruence is a projectivity. A family R which contains the focal surfaces is denoted by R_0. Families R_0 which are attached to certain reduced systems are called "principal".

A number of properties of principal families R_0 are derived. The author then considers the projective invariants of the families R_0 and gives geometric interpretations for some of them. *R. Blum.*

Coxeter, H. S. M. Hyperbolic triangles. Scripta Math. **22** (1956), 5–13.

After a brief historical survey of hyperbolic geometry, the note deals with the following fundamental properties of triangles in the hyperbolic plane: a) Gauss's proof that the area of a triangle is proportional to its angular defect; b) this area remains finite even when the sides are infinitely long; c) a projective proof of the hyperbolic counterpart of the Euclidean (or elliptic) theorem that the six bisectors of the three angles of a triangle are the six sides of a complete quadrangle. *L. A. Santaló.*

Cassina, Ugo. Sulla dimostrazione di Wallis del postulato quinto di Euclide. Period. Mat. (4) **34** (1956), 197–219.

See also: Coxeter, p. 378; Bielecki et Radziszewski, p. 412; Erëmin, p. 417; Ruben, p. 426; Frederick, p. 434; Dusi, p. 452.

Convex Domains, Integral Geometry

Lumer, G. Polygons inscriptible in convex curves. Rev. Un. Mat. Argentina **17** (1955), 97–102 (1956), (Spanish)

Shisha, Oved. A remark on Fejér's theorem on the convex hull of a point-set. Riveon Lematematika **9** (1955), 75–77. (Hebrew. English summary)

In the following lemma, if S^* is the closed convex hull of a point set S in Euclidean n-space and $P \notin S^*$, then some Q is nearer than P to every ϵS [for $n=2$ and compact S in Fejér, Math. Ann. **85** (1922), 41–48], Q can be selected in S^*, e.g. its nearest point to P.
T. S. Motzkin (Los Angeles, Calif.).

Bielecki, Adam; et Radziszewski, Konstanty. Sur les parallélépipèdes inscrits dans les corps convexes. Ann. Univ. Mariae Curie-Skłodowska. Sect. A. **8** (1954), 97–100 (1956). (Polish and Russian summaries)

The authors prove that, in any convex solid of positive volume V, it is possible to inscribe a parallelepiped of volume $v \geq \frac{2}{9}V$. *H. S. M. Coxeter* (Toronto, Ont.).

Bielecki, Adam. Quelques remarques sur la note précédente. Ann. Univ. Mariae Curie-Skłodowska. Sect. A. **8** (1954), 101–103 (1956). (Polish and Russian summaries)

H. Steinhaus, in his "Mathematical snapshots", p. 86 [Oxford, 1950; MR **12**, 44] points out that a square can be inscribed in any closed curve in a plane. No analogous result can be expected in three dimensions; for the author gives an example of a convex solid in which it is impossible to inscribe a rectangular parallelepiped.
H. S. M. Coxeter (Toronto, Ont.).

Pozzolo Ferraris, Giulia. Sopra alcuni problemi geometrici di massimo e minimo. Period. Mat. (4) **34** (1956), 228–233.

A solution by elementary means of certain problems ordinarily solved by the use of the differential calculus.

Yosida, Yôiti. Sur l'inégalité entre les moyennes arithmétique et géométrique. Comment. Math. Univ. St. Paul. **5** (1956), 1–2.

The author gives a proof based upon elementary calculus of the inequality mentioned in the title [for others see, e.g., Hardy, Littlewood, and Pólya, Inequalities, Cambridge, 1934, Ch. IV]. This proof is inductive and makes use of the function

$$[(a_1+a_2+\cdots+a_k+x)/(k+1)]^{k+1}-a_1a_2\cdots a_kx.$$

It is not explicitly complete for $(a_1+a_2)/2 \geq (a_1a_2)^{1/2}$.
J. Aczél (Debrecen).

See also: Černikov, p. 371; Remez, p. 371; Braumann, p. 371; Bremermann, p. 387; Burger, p. 417; Erëmin, p. 417; Kuhn, p. 417; Richter, p. 423; Masuyama, p. 425; Wolfe, p. 449; Mills, p. 450; Debreu, p. 451.

Differential Geometry

Seifert, L. Contributions à la théorie des hélices tracées sur les surfaces de révolution du second degré. Publ. Fac. Sci. Univ. Masaryk **1955**, 387–406. (Czech. Russian and French summaries)

On étudie la courbe

$$x=a(\cos v+v\sin v),\ y=a(\sin v-v\cos v)$$

située sur le paraboloide $x^2+y^2=2pz+p^2$ et les courbes données par l'équation différentielle $dx^2+dy^2=k^2dz^2$ et

situées sur l'ellipsoide $(x^2+y^2)/a^2+y^2/b^2=1$ $(b>a)$. Ces courbes sont les hélices cylindre-coniques. Les invariants différentiels, les projections des courbes et quelques propriétées algébriques sont trouvées. *F. Vyčichlo.*

Löbell, Frank. Zur Konstruktion geschlossener Clifford-Kleinscher Räume negativer Krümmung. Bayer. Akad. Wiss. Math.-Nat. Kl. S.-B. **1955**, 175–185 (1956).

A simplification of an earlier construction of the author's [Ber. Verh. Sächs. Akad. Wiss. Leipzig. Math.-Phys. Kl. 83 (1931), 167–174]. *R. J. Walker.*

Lucaroni, Raffaele. Grafici di curve algebriche con l'uso delle determinatrici di Cramer. Archimede **8** (1956), 272–277.

Amici, Andrea. Studio dell'equazione $x^{y/x}=y^{x/y}$. Ricerca, Napoli **7** (1956), 59–63.

Cîmpan, Florica T. La généralisation de la notion de podaire. Rev. Univ. "Al. I. Cuza" Inst. Politehn. Iaşi **1** (1954), 28–43. (Romanian. Russian and French summaries)

Le podaire C_{12} d'une courbe C_1 par rapport à une courbe C_2; C_{12} est la courbe orthoptique du couple C_1, C_2. Cas où C_1 est une droite. Extension des théorémes de Catalan et Steiner-Raabe pour le podaire ordinaire.

O. Bottema (Delft).

Heinz, Erhard. On the existence problem for surfaces of constant mean curvature. Comm. Pure Appl. Math. **9** (1956), 467–470.

This note is a summary of results obtained by the author in the paper reviewed in MR **16**, 1115.

★Bianchi, Luigi. Opere. Vol. IV. Deformazioni delle quadriche, teoria delle trasformazioni delle superficie applicabili sulle quadriche. Parte seconda. Edizioni Cremonese, Roma, 1956. 366 pp.

The following eleven papers (with titles abbreviated) are included in this volume: transformations des surfaces [Mém. Sav. Etr., (2), **34** (1909)]; caso limite delle trasformazioni [Rend. Acc. Naz. Lincei, (5) **18** (1 sem. 1909)]; trasformazioni de Guichard [Lincei (5), **20** (1 sem. 1911)]; sistemi coniugati permanenti [Lincei, (5), **22** (1913, 2 sem.)]; sistemi tripli coniugati [Lincei, (5), **23** (1914, 1 sem.)]; deformate rigate [Lincei, (5), **23**, (2 sem. 1914)]; sistemi tripli coniugati [Ann. di Mat. pura ed appl., (3), 23 (1914)]; singular transformations [Bull. Amer. Math. Soc. **23** (1917)]; singular transformations [Trans. Amer. Math. Soc., **18** (1917)]; costruzione geometrica di Darboux [Ann. Mat. pura ed appl. (4), **1**, (1923–24)]; costruzione geometrica di Darboux [Rend. Sem. Mat. Roma, (2), **2** (1925)].

Gheorghiev, Gh. Sur la théorie des complexes de droites en geométrie euclidéenne. Acad. R. P. Romîne. Fil. Iaşi. Stud. Cerc. Şti. **6** (1955), 105–113. (Romanian. Russian and French summaries)

The purpose of this paper is to show the connection which exists between the theory of complexes of straight lines and that of a field of unit vectors (F) in the euclidean three-space.

To a point M of (F) one can associate three mutually orthogonal unit vectors I_1, I_2, I_3, of which I_3 is the vector of (F). The components of the vector of instantaneous rotation of the triad I_1, I_2, I_3 can then be linearly expressed in terms of the three pfaffians which define the infinitesimal displacement of M. The coefficients $p_iq_ir_i$ $(i=1, 2, 3)$ thus obtained form a matrix (which the author calls Darboux's matrix). If M is the center of the complex formed by the vectors of (F) and $I_1I_2I_3$ Frenet's triad of the vectorial line through M then we have $q_3=q_1=0$. The remaining coefficients, among which there exists a relation, and a certain additional quantity are equivalent to the seven euclidean invariants of the complex. Geometrical interpretations of these invariants are then given.

The author determines further the relations which exist among the invariants of particular complexes and applies finally these results to linear complexes.

R. Blum (Saskatoon, Sask.).

Miron, R. Quelques problèmes de la géométrie d'un champ de vecteurs unitaires. Acad. R. P. Romîne. Fil. Iaşi. Stud. Cerc. Şti. **6** (1955), 173–183. (Romanian. Russian and French summaries)

The geometrical properties of a field of unit vectors F in the euclidean three-space are investigated with the help of Cartan's method of exterior differential forms. The author defines the notions of normal and geodesic curvature and of geodesic torsion of F. The sum and the product of the extreme values of the geodesic torsion in a point are called "mean" and "total" torsion. By applying Levi-Civita's parallelism it is shown that the mean torsion is a measure of the non-closure of an infinitesimal parallelogram orthogonal to F. A number of other results connected with the mean and total torsion are obtained.

These investigations are closely connected with the work of the paper reviewed above whose notation has been adopted. *R. Blum* (Saskatoon, Sask.).

Kovančov, N. I. Representation without integrals of certain special classes of complexes. Mat. Sb. N.S. **38(80)** (1956), 107–128. (Russian)

Following A. Voss [Math. Ann. **9** (1876), 55–162] and P. Mentré, [C. R. Acad. Sci. Paris **175** (1922), 941–943], the author investigates so-called inflection centers on a line l of a linear complex. To obtain them, consider an arbitrary plane through l and in it a curve enveloped by the lines of the complex. When the point of tangency of l with this curve is singular this point is an inflection center. Its determination depends on a biquadratic equation. It is shown how to construct complexes for which these centers are 1) one double one, two single ones, 2) two double ones, 3) one triple, one single, and when there is 4) one four-fold center. The nature of the theorems may be illustrated by the fourth case: To obtain such a complex with four-fold center take an arbitrary ruled surface Σ and establish an arbitrary elliptical involution between the points A_1, A_2 of a generator; then let A_1 be the center of a line pencil in the tangent plane to Σ at A_2. Then Σ is the locus of the four-fold inflection centers of the line complex formed by these line pencils. *D. J. Struik* (Cambridge, Mass.).

Godeaux, Lucien. Sulle congruenze *W*. Rend. Mat. e Appl. (5) **15** (1956), 36–45.

See also: Riesz, p. 401; Séminaire Henri Cartan, p. 409; Mayer, p. 412; Mangeron, Ciobanu et Braier, p. 427; Mansfield, p. 433; Gotusso, p. 438.

Riemannian Geometry, Connections

Ōtsuki, Tominosuke. Associated Riemannian manifolds and motions. Math. J. Okayama Univ. **5** (1955), 13–42.

The associated Riemannian manifold V_N, $N=\frac{1}{2}n(n+1)$, to a given Riemannian manifold V_n is the bundle of orthogonal frames of V_n, with a metric determined by the metric of V_n. The parameters of the connection and the curvature forms of V_N are expressed in terms of those of V_n. This leads to Theorem 1 which says that V_N is Einstein if V_n is Einstein, and in addition the curvature of V_n satisfies another condition. A corollary says: if V_n is of constant curvature, then V_N is Einstein if and only if $K>(n-2)/2(n-1)$. Another result is that each geodesic in V_N makes a constant angle with the horizontal planes of the connection.

Every motion on V_n leads to a motion of V_N, of course, and so does every right translation of V_n; these two sets of isometries of V_N commute (Theorems 2 and 3). A sequence of motions of V_n converges (tangentially) at every point of V_n, or at no point of V_n (Theorem 4); and the tangent fields thus arising are differentiable and represent infinitesimal motions (Theorem 6). A section on holonomy groups comes at the end.

A. Nijenhuis (Seattle, Wash.).

Murgescu, Viorel. Espaces à connexion affine, à métrique angulaire. Acad. R. P. Romîne. Fil. Iaşi. Stud. Cerc. Şti. **6** (1955), no. 1–2, 185–199. (Romanian. Russian and French summaries)

This is a study of spaces with (in general non-symmetric) affine connection in two dimensions (A_2), which admit the notion of angle $\Omega(x^i, \lambda^i, \mu^i)$ $(i=1, 2)$ of two directions (λ^i, μ^i) satisfying the following condition. I) It is independent of the length of the sides of the angle. Therefore Ω is homogeneous of degree zero in λ^i, μ^i. II) It is invariant under transformations of coordinates in (A_2). III) It is multiplied by a factor, function of x^i, when transported by parallelism. From this follows that $d\Omega=\Omega\cdot\varphi_i dx^i$.

The consideration of the resulting system of linear partial differential equations in the unknown function Ω leads to the following three solutions. a) Spaces (A_2) with absolute parallelism for which

$$\varphi_{ij}=\frac{\partial\varphi_j}{\partial x^i}-\frac{\partial\varphi_i}{\partial x^j}=0 \quad (i, j=1, 2).$$

b) Spaces (A_2) which admit at least one field of parallel directions and

$$\varphi_{ij,k}=a_k\varphi_{ij} \quad (i, j, k=1, 2),$$

where $\varphi_{ij,k}$ is the covariant derivative of φ_{ij}. c) Spaces (A_2) for which the curvature tensor and φ_{ij} and their covariant derivatives satisfy certain (less simple) relations. The function Ω is found in these cases.

This paper generalizes results obtained by A. Haimovici [Com. Acad. R. P. Romîne **1** (1951), 157–163; Acad. R. P. Romîne. Fil. Iaşi. Stud. Cerc. Şti. **2** (1951), 66–82; MR **17**, 408] who considered spaces (A_2) with a function Ω satisfying conditions I, II, and (instead of III) is invariant under parallel transport.

R. Blum.

Ide, Saburo. On the Wirtinger's connections in higher order spaces. J. Fac. Sci. Hokkaido Univ. Ser. I. **13** (1956), 75–119.

W. Wirtinger introduced the so-called Wirtinger connection, having in mind the possibility of applications to the theory of relativity and astronomy [Trans. Cambridge Philos. Soc. **22** (1922), 439–448]. This connection is based on the concept of "double vector" and is a nonlinear connection. A double vector means a pair consisting of a contravariant vector η^α and a covariant one ν_α satisfying the incidence condition $\eta^\alpha\nu_\alpha=0$. After a brief explanation of the Wirtinger connection the author generalizes it to the space of line elements of higher order $(x^\alpha, x^{(1)\alpha}, \cdots, x^{(m)\alpha})$, assuming that ν_α depends not only on (x, η) but also on $(x^{(1)}, \cdots, x^{(m)})$ and then introduces the generalized Wirtinger connection in a Kawaguchi space with the metric $ds=F(x, x^{(1)}, \cdots, x^{(m)})dt$ [cf. Kawaguchi. Proc. Imp. Acad. Tokyo **13** (1937), 237–240]. The absolute differential and the covariant derivatives of a double vector are obtained in detail. Especially, in the special Kawaguchi space with the metric

$$ds=\{A_i(x, x')x''^i+B\}^{1/p}$$

[cf. Kawaguchi. Trans. Amer. Math. Soc. **44** (1938), 153–167] the theory is developed more extensively, making use of the fact that x' and A_i form a double vector. Many results on the curvature tensors and various identities in the special Kawaguchi space are obtained, e.g. two kinds of curvature tensors corresponding to the two connections C and C' are made clear.

A. Kawaguchi.

Varga, O. Eine Charakterisierung der Kawaguchischen Räume metrischer Klasse mittels eines Satzes über derivierte Matrizen. Publ. Math. Debrecen **4** (1956), 418–430.

The author considers a Kawaguchi space such that in the n-dimensional space the area element of a p-dimensional surface is defined by

$$dO=L(x^1, \cdots, x^n; [dx^{i_1}\cdots dx^{i_p}])$$

depending on components $[dx^{i_1}\cdots dx^{i_p}]$ of the tangent simple p-vector at the point x of the surface, where the function $L(x; q)$ of n variables x^i and of $N\left(=\binom{n}{p}\right)$ variables $q^{i_1\cdots i_p}$, which may not be zero at the same time and may be taken as components of a (not necessarily simple) p-vector, is assumed to be positive always, continuously differentiable with respect to its all arguments sufficiently many times and positively homogeneous of dimension one in the arguments q's [cf. Kawaguchi, Monatsh. Math. Phys. **43** (1936), 289–297]. Differentiating twice $\frac{1}{2}L^2$ by q's, we obtain the tensor $g_{i_1\cdots i_p;j_1\cdots j_p}$ which is skew-symmetric with respect to the indices i's and j's respectively. The Kawaguchi space is called of metric class, when the relation

$$g_{i_1\cdots i_p;j_1\cdots j_p}=p!g_{[i_1[j_1}g_{i_2j_2}\cdots g_{i_p]j_p]}$$

holds, i.e. the quadratic regular matrix of Nth order formed by g's is the pth derived one of a matrix of nth order [cf. R. Debever, Thèse, Bruxelles, 1947, pp. 1–39; MR **9**, 379]. The author gives a slightly modified definition to a Kawaguchi space of metric class, but it is essentially the same as the above-mentioned. And he finds a necessary and sufficient condition for a Kawaguchi space to be of metric class. The condition consists of four kinds of relations among the components of $g_{i_1\cdots i_p j_1\cdots j_p}$. The method of proof used by the author is purely algebraic. {Reviewer's remark: Another kind of necessary and sufficient condition for the same matter is stated in the paper by the reviewer and K. Tandai [Tensor (N.S.) **2** (1952), 47–58; MR **14**, 586].}

A. Kawaguchi (Sapporo).

Su, Buchin. On the isomorphic transformations of minimal hypersurfaces in a Finsler space. Acta Math. Sinica 5 (1955), 471–488. (Chinese. English summary)

Let $\Gamma_{ikh}{}^*$ be the affine connection in a Finsler space introduced by W. Barthel [Ann. Mat. Pura Appl. (4) 36 (1954), 159–190; MR 16, 173] such that

$$\Gamma_{ikh}{}^*-\Gamma_{hik}{}^*=l_iA_k|_h-l_hA_k|_i.$$

A hypersurface in such a space is said to be minimal if $M\equiv G^{\rho\sigma}a_{\rho\sigma}\equiv(g^{\rho\sigma}+A^\rho A^\sigma+x_k{}^\rho A^k\|^\sigma)a_{\rho\sigma}=0$. A necessary and sufficient condition is obtained for the infinitesimal deformation $\bar{x}^i=x^i+\xi^i(x)\delta t$ to deform a minimal hypersurface S to a neighboring minimal hypersurface $\bar{S}$. From this condition there is deduced an invariant which is analogous to the invariant U in Barthel's paper and depends on $G^{\rho\sigma}$ and the curvature tensor $R_{\rho0\sigma0}$.

C. C. Hsiung (Bethlehem, Pa.).

See also: Bilby and Smith, p. 430.

Complex Manifolds

Guggenheimer, Heinrich. Sur la définition des genres d'une variété complexe non kählérienne. Boll. Un. Mat. Ital. (3) 11 (1956), 328–331.

Let V be a paracompact complex manifold, $W\subset V$ be a complex submanifold, and (V', W') be a modification of (V, W) [Guggenheimer, Ann. Mat. Pura Appl. (4) 41 (1956), 87–93; MR 17, 1239]. The author shows, by an exact sequence argument, that $H_{d''}{}^{v,k}(V)\cong H_{d''}{}^{v,k}(V')$ $(k=0, 1, \cdots, v)$, where $v=\dim V=\dim V'$, and $H_{d''}{}^{v,k}(V)$ is the d''-cohomology of exterior differential forms of type (v, k) on V. Moreover, if V is compact and $p_i(W)=$ rank $H_{d''}{}^{i,0}(W)$, then

$$\sum_{i=0}^{w}(-1)^ip_i(W)=\sum_{i=0}^{w}(-1)^ip_i(W').$$

R. C. Gunning (Princeton, N.J.).

Algebraic Geometry

Spampinato, Nicolò. Il genere di una curva algebrica di S_r in funzione del carattere cuspidale e dell'ordine completo della curva. Giorn. Mat. Battaglini (5) 4(84) (1956), 145–149.

Spampinato, Nicolò. Relazione fra le proprietà di contatto e di osculazione di una curva o superficie con le algebre dei numeri biduali, triduali e tripotenziali. Ricerca, Napoli 6 (1955), no. 4, 3–10 (1956).

Spampinato, Nicolò. La V_5 di S_8 determinata dalle coniche osculatrici di una curva algebrica piana prolungata nel campo tripotenziale. I. Ricerca, Napoli 7 (1956), 43–58.

Godeaux, Lucien. Sur une surface du cinquième ordre possédant une droite double tacnodale. I, II. Acad. Roy. Belg. Bull. Cl. Sci. (5) 42 (1956), 884–896, 897–905.

Gherardelli, Francesco. Alcune osservazioni sui sistemi canonici e pluricanonici di una varietà algebrica. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 19 (1955), 28–32.

On a surface $F:f(x, y)=0$, in S_3, with ordinary singularities, let $\omega^{(r)}=\sum_0^r a_i(dk)^i(dy)^{r-i}$ be a differential form of dimension one, of first kind and of order $r>1$, and with discriminant Φ not identically zero. $\omega^{(r)}=0$ defines a system Σ of curves on F, and $\Phi=0$ the envelope I of Σ. If D is the double curve of F and J the branch curve of the function $z(x, y)$, it is shown that $I+3r(r-1)\equiv r(r-1)J$. Since $J\equiv 3D+K$, K being a canonical curve, this gives $I\equiv r(r-1)K$. Similarly, if T is the curve of contact of systems defined by $\omega^{(r)}=0$ and $\omega^{(s)}=0$ we get $T+3rsD\equiv rsJ$ and $T\equiv rsK$. An application of these results is a quick derivation, in the case $q\geq 3$, of the relation $2\pi-2=48(P_a-1)+(I-6)$ for the genus π of a canonical curve of a V_3 of irregularity q, arithmetic genus P_a, and Zeuthen-Segre Invariant I.

R. J. Walker (Ithaca, N.Y.).

Bilo, J. On the Lüroth varieties of a linear n-space. Simon Stevin 31 (1956), 31–36.

This is an expository note which includes a generalization of Pascal's theorem on conics to normal rational curves of S_n.

H. T. Muhly (Iowa City, Ia.).

Engel, Wolfgang. Invariante Divisorenscharen bei endlichen Gruppen von Cremonatransformationen. J. Reine Angew. Math. 196 (1956), 59–66.

It is proved that every finite group of Cremona transformations of a plane into itself admits an invariant linear system of curves of genus 0 or 1, and the different possibilities for such a linear system are classified. The results are arrived at by the methods of ideal theory, and ought to be compared with old ones obtained geometrically by A. Wiman [Math. Ann. 48 (1897), 195–240] and others.

B. Segre (Rome).

Godeaux, Lucien. Remarques sur la formation des systèmes canoniques et pluricanoniques de quelques surfaces algébriques. Acad. Roy. Belg. Bull. Cl. Sci. (5) 42 (1956), 1002–1011.

See also: Manin, p. 380.

NUMERICAL ANALYSIS

Numerical Methods

Zajta, A. Untersuchungen über die Verallgemeinerungen der Newton-Raphsonschen Wurzelapproximation. Acta Tech. Acad. Sci. Hungar. 15 (1956), 233–260. (Russian, English and French summaries)

Newton's iteration $x_{n+1}=x_n-f(x_n)/f'(x_n)$, $\lim x_n=\alpha$, α a root of $f(x)=0$, has been generalized in many directions. The present paper is based on a process described by Kiss [Z. Angew. Math. Mech. 34 (1954), 68–69; MR 15, 900] in which the factor $(f')^{-1}$ is replaced by other suitably chosen rational functions of $f, f', \cdots, f^{(n)}$ (taken for x_n) and which lead to approximations of higher order than the second order Newton algorithm (approximation of order n when $f(\alpha)=f'(\alpha)=\cdots=f^{(n-1)}(\alpha)=0$, $f^{(n)}(\alpha)\neq 0$). In examples considered by Kiss only the first step of approximation occurs, represented by Zajta in the form (1) $x_{n+1}=a-f(a)\cdot K_{n-1}/K_n$, K_n a polynomial of degree n in $f(a), f'(a), \cdots, f^{(n)}(a)$. Thus x_{n+1} is the first, not the nth, step of the algorithm. For $K_0=1$, $K_1=f'$, we have Newton; for $K_2=f'^2-ff''/2$ a formula of order 3; for K_3, K_4 two Kiss formulae of order 4, 5, respectively.

Zajta extends K_n to the case of general n with (1) representing an approximation of order $n+1$, and examines in detail the structure of the polynomials $K_n(f, f', \cdots, f^{(n)})$. He derives an explicit formula for K_n which is mainly of theoretical interest, due to its complicated structure. It is apparently overlooked that K_n is representable as a determinant of order n; $K_n=|a_{ik}|$; $a_{ik}=f^{(i-k+1)}/(i-k+1)!$ $1\leq i, k\leq n$; $f^{(0)}=f$; $f^{(\mu)}=0$, $\mu<0$.

If n can be made to grow infinite, preserving convergence, the algorithm will yield in one step the exact root of $f(x)=0$. This line of investigation is pursued in a formal manner. *A. J. Kempner* (Boulder, Colo.).

Salzer, Herbert E. Formulas for the partial summation of series. Math. Tables Aids Comput. **10** (1956), 149–156.

Tables are given for the coefficients $A_m(n)$, for use in evaluating an expression $S_n\sim\sum_{m=4}^{10} A_m(n)S_m$, an approximation to the nth term of a slowly convergent sequence whose values are known for $n=4(1)10$. The choice of the numbers 4 and 10 is apparently determined by experience and convenience.

The formula is effectively Lagrangian extrapolation of S_m considered as a polynomial in m^{-1}, and the coefficients come from the expression

$$A_m(n)=\frac{m^6}{n^6(n-m)}\prod_{j=4}^{10}(n-j)/\prod_{j\neq m}(m-j).$$

These quantities are written in the form $C_m(n)/D(n)$, and C and D are tabulated as exact integers for

$$n=11(1)50(5)100(10)200(50)500(100)1000.$$

Except for $n=47$ the number D has at most ten digits, other than final zeros, which facilitates division on a ten-bank calculating machine.

Several illustrative examples, some using auxiliary sequences to ensure convergence, indicate the power and accuracy of the method. The paper is an extension of the author's previous work [J. Math. Phys. **33** (1955), 356–359; MR **16**, 863] on "complete" summation.

L. Fox (Berkeley, Calif.).

Chisnall, G. A. A modified Chebyshev-Everett interpolation formula. Math. Tables Aids Comput. **10** (1956), 66–73.

Everett's interpolation formula is rearranged in the form

$$f_p=qf_0+C_3(q)\beta^2f_0+C_5(q)\beta^4f_0+\cdots+pf_1+C_3(p)\beta^2f_1+C_5(p)\beta^4f_1+\cdots,$$

where $q=1-p$, $C_n(q)=\cos\{n\cos^{-1}(q\cos\pi/2n)\}$, and $\beta^{2s}f$, a "modified difference" of order $2s$, is a linear combination of ordinary even differences $\delta^{2s}f$, $\delta^{2s+2}f$, $\cdots$.

Only a few terms of the rearranged formula are needed for high accuracy in interpolation, in accordance with the rapidly convergent properties of Chebyshev series, and modified differences would be included in the table for the benefit of the user. The method is not new in principle, but this is the first full presentation of the theory, and exact equations are given from which the coefficients of $\delta^{2s}f$, $\delta^{2s+2}f$, $\cdots$, comprising $\beta^{2s}f$, can be calculated. Numerical values are also given, to twelve decimals, for coefficients of the first ten modified differences.

The author notes the existence of a similar rearrangement of Bessel's formula, but gives no results. {An example indicating the superiority of the Everett rearrangement over that of Bessel is inconclusive since the Bessel formula chosen is not the natural analogy of the Everett rearrangement. This reviewer has found that the corresponding Chebyshev-Bessel formula, transformed to Everett form, with slightly different numerical coefficients in the modified differences, has in fact a slightly smaller maximum error in most practical cases of interpolation}. *L. Fox* (Berkeley, Calif.).

Thompson, Philip Duncan. Optimum smoothing of two-dimensional fields. Tellus **8** (1956), 384–393.

This paper deals with the smoothing out of nonsystematic errors in observations — a problem of increasing significance as methods of numerical weather prediction improve. The author tackles the problem in a manner similar to that used in communication theory [see Wiener, Extrapolation, interpolation, and smoothing of stationary time series, Wiley, New York, 1949; MR **11**, 118], and finds a weighting function such that the RMS difference between the true field and the smoothed field of observations is a minimum; this is the condition for optimum smoothing. By taking a synthetic field of observations, consisting of an artificially constructed error field superimposed on a specified true field, a comparison is made of the results of optimum smoothing with those of subjective smoothing practiced by an experienced synoptic analyst. The results indicate that the process of optimum smoothing is at least as good as that of subjective smoothing. Finally, the author estimates the information value of a network of observations which are subject to nonsystematic error. *M. H. Rogers.*

Michalup, Erich. Entwicklung und Weiterbildung von mechanischen Ausgleichungsformeln. Statist. Vierteljschr. **9** (1956), 64–69.

Brachman, Malcolm K. Notes on the summation of series. J. Soc. Indust. Appl. Math. **3** (1955), 254–258.

Certaines sommes finies particulières dont les termes s'expriment au moyen de coefficients du binôme peuvent être exprimées en utilisant l'artifice suivant

$$\text{symbole de Kronecker } \delta_{m,n}=\frac{1}{2j\pi}\oint\frac{d\lambda}{\lambda^{m-n+1}}.$$

J. Kuntzmann (Grenoble).

Crowder, H. K. On the evaluation of certain finite power series. Ordnance Computer Research Report, Ballistic Research Laboratories, Aberdeen Proving Ground, Md., vol. 3 (1956), no. 3, pp. 1–2. (Government Agencies, their contractors and others cooperating in Government research may obtain reports directly from the Ballistic Research Laboratories. All others may purchase photographic copies from the Office of Technical Services, Department of Commerce, Washington 25, D.C.)

This paper discusses some well-known methods of handling the evaluation of polynomials where the ratios of consecutive terms are simple. *R. W. Hamming.*

★ **Hoffman, A. J.; and Kuhn, H. W. On systems of distinct representatives.** Linear inequalities and related systems, pp. 199–206. Annals of Mathematics Studies, no. 38. Princeton University Press, Princeton, N. J., 1956. $5.00.

Let S_1, S_2, $\cdots$, S_n be subsets of a set S; and let T_1, T_2, $\cdots$, T_p be mutually disjoints subsets of S with union equal to S. Let $n+p$ be finite. Let integers c_k, d_k ($k=1$,

2, $\cdots$, p) be given such that $0 \leqq c_k \leqq d_k$. Let the cardinal of a set A be denoted by $|A|$. Necessary and sufficient conditions are given for the existence of a set R of n distinct elements $a_1, a_2, \cdots, a_n$ such that (i) $a_i \in S_i$ ($i=1, 2, \cdots, n$) and (ii) $c_k \leqq |R \cap T_k| \leqq d_k$. These conditions are that

$$|(\bigcup_{j \in A} S_j) \cap (\bigcup_{k \in A} T_k)| \geqq |A| - n + \sum_{k \in B} c_k,$$
$$|(\bigcup_{j \in A} S_j \cap (\bigcup_{k \notin B} T_k) \geqq |A| - \sum_{k \in B} d_k$$

for all subsets A of (1, 2, $\cdots$, n) and all subsets B of (1, 2, $\cdots$, p). The necessity is easy to see; the sufficiency is proved with the help of the duality theorem of linear programming. The result represents a considerable generalisation of earlier work on systems of distinct representatives, e.g. that of H. B. Mann and H. J. Ryser [Amer. Math. Monthly **60** (1953), 397–401; MR **14**, 1053].

P. Hall (Cambridge, England).

Černikov, S. N. Positive and negative solutions of systems of linear inequalities. Mat. Sb. N.S. **38(80)** (1956), 479–508. (Russian)

Proof of results announced in Dokl. Akad. Nauk SSSR (N.S.) **99** (1954), 913–916 [MR **17**, 133] with many variants [cf. Motzkin, Thesis, Basel, Jerusalem, 1936, Th. D5]. *T. S. Motzkin* (Los Angeles, Calif.).

Erëmin, I. I. On some properties of nodes of a system of linear inequalities. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 2(68), 169–172. (Russian)

Let P_j and H_j be the hyperplane and halfspace $\sum_k a_{jk}x_k =$ respectively $\leqq a_j$, $\sum_k a_{jk}^2 = 1$. Let

$$C = H_1 \cap \cdots \cap H_m$$

be a nonempty convex polyhedral set and

$$F = C \cap P_{j_1} \cap \cdots \cap P_{j_r}$$

a least-dimensional face of C. For any r independent columns of (a_{jk}), let

$$\Delta_j = \begin{vmatrix} a_{j_1 i_1} \cdots a_{j_1 i_r}\ a_{j_1} \\ \cdot\ \cdots\cdots\ \cdot \\ a_{j_r i_1} \cdots a_{j_r i_r}\ a_{j_r} \\ \cdot\ \cdots\cdots\ \cdot \\ a_{j i_1} \cdots a_{j i_r}\ a_j \end{vmatrix}$$

and the "node" Δ its first r by r minor. Then $\Delta_j/\Delta \geqq 0$ is the distance of C from H_j. *T. S. Motzkin.*

Burger, E. Über homogene lineare Ungleichungssysteme. Z. Angew. Math. Mech. **36** (1956), 135–139. (English, French and Russian summaries)

Detailed algebraic proofs of four theorems constituting the double description method [Motzkin, Raiffa, Thompson and Thrall, Contributions to the theory of games, v. 2, Princeton, 1953, pp. 51–73; MR **15**, 638].

T. S. Motzkin (Los Angeles, Calif.).

Kuhn, H. W. Solvability and consistency for linear equations and inequalities. Amer. Math. Monthly **63** (1956), 217–232.

Exposition, in its logical context, of the well-known algebraic criteria for dependence of a linear relation (equation of inequality) R on others, in particular of a solutionless R (inconsistency), together with a description of the process of elimination used in the proof.

T. S. Motzkin (Los Angeles, Calif.).

★ **Young, David. On the solution of linear systems by iteration.** Proceedings of Symposia in Applied Mathematics. Vol. VI. Numerical analysis, pp. 283–298. Published by McGraw-Hill Book Company, Inc., New York, 1956 for the American Mathematical Society, Providence, R. I. \$9.75.

This article, covering a lecture given at the American Mathematical Society symposium on numerical analysis held at Santa Monica in August 1953, is mostly an exposition of known relations among iterative methods for solving linear systems. It is mainly based on theoretical and experimental work of the author and his collaborators Lerch and Warlick.

Let A be a non-singular square matrix of real elements with all $a_{i,i} > 0$. The basic problem is to solve the system (*) $Ax = d$, where x and d are column vectors. In Richardson's method for solving (*), abbreviated I.m, one forms the sequence $\{x^{(n)}\}$ by:

$$x^{(n+1)} = x^{(n)} + \beta_{n+1}(Ax^{(n)} - d),$$

where m values of the β_i are used in cyclic order. Let $A = B + C + D$, where B, C, D are respectively the diagonal, below-diagonal, and above-diagonal parts of A. In Jacobi's method for solving (*), abbreviated II, one forms the sequence $\{x^{(n)}\}$ by:

$$Bx^{(n+1)} = -Cx^{(n)} - Dx^{(n)} + d.$$

In the successive overrelaxation method of Frankel [Math. Tables Aids Comput. **4** (1950), 65–75; MR **13**, 692] and Young [Trans. Amer. Math. Soc. **76** (1954), 92–111; MR **15**, 562], here abbreviated III.ω, one defines $\{x^{(n)}\}$ by

$$(\omega^{-1}B + C)x^{(n+1)} + [(1-\omega^{-1})B + D]x^{(n)} = d.$$

The author considers four conditions on A: (i) A is symmetric and positive definite; (ii) all $a_{i,j} \leqq 0$ ($i \neq j$); (iii) A is "irreducible" with dominant diagonal; (iv) A has property (A) defined by Young [op. cit.]. He lists many convergence implications among methods I.m, II, and III.ω under various conditions (i)–(iv).

Defining a "rate of convergence," he compares the methods again. For example, when (i) and (iv) hold and the parameters β_i and ω are chosen optimally, III.ω converges roughly twice as fast as I.m for $m \sim \infty$. Numerical experiments on ORDVAC with the solution of Dirichlet's problem over a 19×19 square net are summarized to support the applicability of the theory. There are 26 references.

{Reviewer's remark: On page 283 the author conveys the wrong impression that about $N!$ multiplications are necessarily required to evaluate a determinant of order N. It needs only about $N^3/3$.} *G. E. Forsythe.*

Gouarné, René. Méthode des polygones, méthode des systèmes partiels. Calcul des déterminants et des polynomes caractéristiques des matrices. J. Rech. Centre Nat. Rech. Sci. **7** (1956), 81–89.

The author expounds the "method of polygons" devised by I. Samuel [C. R. Acad. Sci. Paris **229** (1949), 1236–1237] and a new "method of partial systems" for evaluating the determinant and characteristic polynomial of a matrix with a relatively few non-zero elements. The former method, devised for symmetric matrices with zero diagonal terms, is here extended to general matrices. Special notations are introduced to represent the structure of various permutations of the integers 1, $\cdots$, n.

It is shown how to apply the method to determining the secular equation for the normal modes of vibration of certain molecules. *G. E. Forsythe* (Los Angeles, Calif.).

Kosko, E. Reciprocation of triply partitioned matrices. J. Roy. Aero. Soc. **60** (1956), 490–491.

This paper describes an improvement of W. J. Duncan's method [same J. **60** (1956), 131–132] of computing the inverse of a partitioned matrix. T^{-1} in equation (5) is a misprint for J^{-1}. *D. E. Rutherford* (St. Andrews).

von Holdt, Richard Elton. An iterative procedure for the calculation of the eigenvalues and eigenvectors of a real symmetric matrix. J. Assoc. Comput. Mach. **3** (1956), 223–238.

The author proposes an iteration method for the solution of the eigenvalues and eigenvectors of a real symmetric matrix. The paper is a very timely one since in practice there arise many physical problems containing the solution of a large number of simultaneous equations. In contrast with the most prevalent iteration scheme [Frazer, Duncan, and Collar, Elementary matrices···, Cambridge, 1946; MR **8**, 365], which at times is slow, Dr. Von Holdt claims to improve the general convergence. Essentially the method is of a cubic convergence type and should thus insure a much faster end result than the F-D procedure. The amount of labor for the author's method necessitates the use of a high speed calculator for accuracy. *H. Saunders.*

Laasonen, Pentti. Simultane Bestimmung mehrerer Eigenwerte mittels gebrochener Iteration. Ann. Acad. Sci. Fenn. Ser. A. I. no. **217** (1956), 8 pp.

The author considers the definite and self-adjoint eigenvalue problem of Bliss, namely

$$du(x)/dx=[F(x)+\lambda G(x)]u(x);$$

$A\cdot u(a)+B\cdot u(b)=0$ with u as a column-vector of n components and with A, B, F, G as $n\times n$ matrices; A, B do not depend on x. The problem admits a nonsingular $n\times n$ matrix $T(x)$ which satisfies $dT/dx+TF+F'T=0$, $TG+G'T=0$ and $AT^{-1}(a)A'=BT^{-1}(b)B'$. It is aimed to evaluate eigenvalues and eigenvectors by an iteration method due to H. Wielandt. The iteration starts with approximations $\bar\lambda_1, \bar\lambda_2, \cdots, \bar\lambda_m$ to the first m eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_m$ as well as with approximations $v_1, v_2, \cdots, v_m$ to the first m eigenvectors. It is assumed that these eigenvalues are single, that furthermore $\delta_k=\lambda-\bar\lambda_k$ satisfies $0<|\delta_\mu|<|\lambda_\nu-\lambda_\mu|$; $\mu\neq\nu$; $\mu, \nu=1, 2, \cdots, m$; and that v_k admits a nonvanishing scalar product with the kth eigenvector. The iteration is defined by

$$dv_\mu^{(l)}/dx=(F+\bar\lambda_\mu G)v_\mu^{(l)}+Gv_\mu^{(l-1)},$$

where $v_\mu^{(0)}=v_\mu$ and where $v_\mu^{(l)}$ has to satisfy the boundary conditions. The matrices $V^{(l)}=(v_1^{(l)}, v_2^{(l)}, \cdots, v_m^{(l)})$ together with $S=T'G$ are used to set up constant matrices $Q^{(k,l)}=\int_a^b V^{(k)\prime}\cdot S\cdot V^{(l)}dx$; the ordinary matrix-eigenvalue problem $(Q^{(k,l)}-\kappa\cdot Q^{(k+1,l)}\cdot t=0$ furnishes eigenvalues $\kappa_1, \kappa_2, \cdots, \kappa_m$ which are taken as approximations to the differences δ_μ. They converge towards these differences with $k, l\rightarrow\infty$. *H. F. Bückner* (Schenectady, N.Y.).

Pines, Samuel. Iteration in semidefinite eigenvalue problems. J. Aero. Sci. **23** (1956), 380–381.

The author describes a method for finding eigen vectors and characteristic frequencies for a system of equations involving both elastic and rigid body modes. The equations are of the form

$$(C-w^2M)x=0,$$

where M is positive definite of order and rank n and C is semidefinite of order n and rank $n-r$. These equations can be replaced by equivalent equations with the same M but a new C of rank n. Then we may write

$$(I(1/w^2)-C^{-1}M)x=0,$$

from which the iteration is carried out as usual. The author uses the same procedure for building in the orthogonality conditions into the matrix. *W. E. Milne.*

Kron, Gabriel. Solution of complex nonlinear plastic structures by the method of tearing. J. Aero. Sci. **23** (1956), 557–562.

"The purpose of the present paper is to extend the piece-wise numerical solution of complex elastic structures to nonlinear problems also. As a first step in that direction a statically indeterminate truss is assumed to be stressed beyond the proportional limit, and the effect of the plastic stress range is investigated ···. The nonlinear plastic deformation study in the present pages follows the basic formulas of Wilder [NACA Tech. Note no. **2886** (1953)] and Hoff [Quart. Appl. Math. **12** (1954), 49–55, Kron dates this 1944; MR **15**, 760]···. The present paper should not be considered as a matric reformulation of the scalar equations of Wilder. Rather the paper replaces the energy relations of Wilder by the topological concept of 'connection tensor' $\bar{C}$. This latter in turn enables a systematic tearing apart and piecewise solution of complex structures, linear or nonlinear···. It should also be emphasized that the method of tearing merely speeds up and enlarges the scope of whatever conventional method is available to solve a nonlinear problem. If ··· the method of approximation to be used does not converge, ··· the mere fact of tearing ··· will not make the problem convergent···. Due to the very fact that the method of tearing employs the physical system itself or its model, many opportunities arise to make appropriate changes in the model itself (besides tearing) to force a speedier convergence." (Excerpts from the author's introduction, with italics omitted.) *G. E. Forsythe.*

Rutishauser, Heinz. Eine Formel von Wronski und ihre Bedeutung für den Quotienten-Differenzen-Algorithmus. Z. Angew. Math. Phys. **7** (1956), 164–169.

This is another in the author's series of articles on his quotient-difference (QD) algorithm. In paragraph 6 of Rutishauser, same Z. **5** (1954), 496–508 [MR **16**, 863], the author showed how to find the zeros of a polynomial $N(z)$ of degree n by the QD algorithm. To get started it was necessary to develop $f(z)=N_1(z)/N(z)$ in an S continued fraction, where $N_1(z)$ is an arbitrary polynomial of degree $n-1$. It is now shown that if one selects $N_1(z)=z^{n-1}$, the continued fraction expansion can be eliminated. The theoretical background involves a formula due to Wronski for inverting a power series. There is a numerical example. *G. E. Forsythe* (Los Angeles, Calif.).

Wynn, P. On a cubically convergent process for determining the zeros of certain functions. Math. Tables Aids Comput. **10** (1956), 97–100.

Cubically convergent approximation processes for finding roots of an equation $\phi(x)=0$ are known, but they are frequently not used because they involve computation of $\phi''(x_n)$. The author notes that this computation

is not necessary when $\phi(x)$ satisfies a differential equation of the form $p(x)\cdot\phi''+g(x)\cdot\phi'+r(x)\cdot\phi=s(x)$.

The case of complex roots of $\phi(x)=0$ is considered.

A. J. Kempner (Boulder, Colo.).

Orloff, Constantin. Sur la méthode de Graeffe. C. R. Acad. Sci. Paris **243** (1956), 1269–1270.

This note represents an elementary application of Petrovich's "Spectrum theory" [Les spectres numériques, Gauthier-Villars, Paris, 1919] to the solution of polynomial equations with rational integral coefficients. In Graeffe's method of solving $f_0(x)=\sum a_k x^{n-k}=0$ we form $f_0(x)\cdot f_0(-x)=f_1(x)=\varphi_1(x^2)$, and then $f_m(x)=\varphi_m(x^{2m})$, m a given positive integer, by iteration. The arithmetical operations are simplified by use of the spectrum $S_0=f_0(10^h)$, h a suitably chosen positive integer. Let $\bar{S}_0=(-1)^n\cdot f_0(-10^h)$, then $S_1=S_0\cdot\bar{S}_0$ is the spectrum of $f_1(x)$. From S_1 the polynomial $f_1(x)$ is read off, and the process continued to determine $f_m(x)$. The resulting reduction in the number of arithmetical operations required is examined.

It is stated that the idea is applicable to the computation of zeros of transcendented functions possessing a Taylor expansion. *A. J. Kempner.*

Mikeladze, Š. E. Formulas of mechanical quadratures for multiple integrals. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze **22** (1956), 277–299. (Russian)

The special results cannot be summarized in a short space but the method is of interest. The relation to multiple integrals is incidental and occurs through the formula

$$\int_{x_0}^{x_1}\int_{x_0}^{\tau}\cdots\int_{x_0}^{\tau} f(\tau)d\tau^k=\frac{h^k}{(k-1)!}\int_{-1}^{1}(1-t)^{k-1}f(a+th)dt,$$

where $a=\frac{1}{2}(x_0+x_1)$, $h=\frac{1}{2}(x_1-x_0)$. After stating this the author considers integrals such as $\int_b^a \mathfrak{p}(x)f(x)dx$, first with finite limits, then with one or both limits infinite. In the latter cases $\mathfrak{p}$ is taken to contain an exponential factor.

For a and b finite, the endeavor is to express the integral either, in the usual way, as a linear combination of the values of f at selected points, or else as a linear combination of its divided differences, and in either case to the highest possible order of accuracy. The method stems from the observation that if $x_1, x_2, \cdots, x_m$ are zeros of a polynomial $P_m(x)$ of degree m of the set of polynomials orthogonal with respect to the weight $\mathfrak{p}(x)$, and if $f(x)$ is expanded as a series in its divided differences

$$f(x_1), f(x_1, x), \cdots, f(x_1, x_2, \cdots, x_n), f(x_1, x_2, \cdots, x_m, x_1), \cdots,$$

then when the series is substituted into the integral certain terms disappear as a result of the orthogonality. A simple extension of the notion permits treatment of cases where a limited number of relations are imposed upon the x_i, e.g. that certain ones are fixed.

A. S. Householder (Madison, Wis.).

Salzer, Herbert E. Osculatory extrapolation and a new method for the numerical integration of differential equations. J. Franklin Inst. **262** (1956), 111–119.

Certain advantages are suggested in using for extrapolation and checking the Hermite interpolation formula

$$(1)\quad f(x_0+nh)=\sum_{k=0}^{n-1}[C_k f(x_0+kh)+hD_k f'(x_0+kh)]+R_{2n},$$

where $R_{2n}=f^{(2n)}(\xi)h^{2n}F_n$, $x_0<\xi<x_0+nh$ [see Salzer, J. Res. Nat. Bur. Standards **52** (1954), 211–216; MR **15**, 830, 1140], as compared with classical finite difference formulas involving ordinates only, and numerical values of C_k, D_k and F_n are given for $n=2, 3, \cdots, 11$. A procedure is described for stepwise numerical integration of a differential equation of the form $y'=\phi(x, y)$ in which an extrapolated value of y is obtained by formula (1), the corresponding value of y' is computed from the differential equation, and the value of y is then improved by the refining formula

$$(2)\quad f(x_0+nh)=\sum_{k=0}^{n-1}A_k f(x_0+kh)+h\sum_{k=0}^{n}B_k f'(x_0+kh)+R_{2n+1},$$

where the summation terms in the right member represent the polynomial of degree $2n$ having the same ordinates as $f(x)$ for $x=x_0+kh$, $k=0, 1, \cdots, n-1$ and the same first derivatives for $k=0, 1, \cdots, n$. Numerical values of A_k and B_k are given for $n=1$ to 5. Application of the method over a Cartesian grid in the complex plane is also described and illustrated. {Reviewer's note: The remainder term in formula (2) is not given, but is easily derivable from a result of Kunz [see the paper reviewed above] and is $R_{2n+1}=f^{(2n+1)}(\xi)h^{2n+1}\ E_n$, where

$$x_0<\xi<x_0+nh,$$

$$E_n=-(n!)^2B_{n+1}/(2n+1)!,\ B_{n+1}=[2\sum_{i=1}^n i^{-1}]^{-1}.$$

Correction: In formula (3) read $x_{[n/2]+1}$ for $x_{[n/2]}$.}

T. N. Greville (Washington, D.C.).

Kunz, K. S. High accuracy quadrature formulas from divided differences with repeated arguments. Math. Tables Aids Comput. **10** (1956), 87–90.

The author derives a class of linear numerical quadrature formulas, possibly applicable to the multi-step numerical integration of ordinary differential equations, of the form

$$\sum_{p=0}^{n}A_{np}f(x_p)=h\sum_{p=0}^{n}B_{np}f'(x_p)+O(h^{2n+1}),$$

where h is the interval size, x_p the points of integration, and $f'(x)$ the function being integrated over $n+1$ points. He gives tables for A_{np} and B_{np} for n ranging from 1 to 6 and p from 0 to 6. This class of formulas is a special case of a more general expression derived, which gives general linear relationships between the values of a function and its derivatives of higher order at a finite set of points, in terms of certain high-order divided differences.

{The author proposes the use of such formulas in "checking the accuracy of integration in the numerical solution of differential equations." Today, however, a vast majority of such integrations are being done on stored program digital computers, where such checking procedures are obviated. More important to the possible user would be a discussion of the stability of the various procedures as to error growth under arbitrary round-off and truncation errors. Their practical application as multi-step techniques for numerical integration must therefore await further investigation.}

J. W. Carr, III (Ann Arbor, Mich.).

Brazma, N. A. On solution by the method of nets of a very simple mixed problem for the matrix telegraph equation. Latvijas PSR Zinātņu Akad. Vēstis **1956**, no. 3 (104), 133–138. (Russian. Latvian summary)

Using a finite difference method the author proves the existence and uniqueness of a generalized solution for

the telegraph equation

$$(1)\qquad \frac{\partial^2 u}{\partial x^2} - A_1 \frac{\partial^2 u}{\partial t^2} - A_2 \frac{\partial u}{\partial t} - A_3 u - f = 0,$$

where A_i are given variable $n\times n$ matrices, u an unknown column matrix and f a given column matrix. The solution is sought for in the domain $z\leq x\leq l$, $z\leq t\leq T$ under the boundary conditions

$$(2)\qquad u(0, t)=0,\ u(l, t)=0,$$

$$(3)\qquad u(x, 0)=\phi(x),\ u'_t(x, 0)=\psi(x).$$

It is assumed that the matrix S_1 may be transformed into a diagonal matrix with positive elements. The method follows closely the one proposed by O. A. Ladyženskaya in her book "The mixed problem for a hyperbolic equation" [Gostehizdat, Moscow, 1953; MR **17**, 160].

L. Bers (New York, N.Y.).

Goodman, T. R.; and Lance, G. N. The numerical integration of two-point boundary value problems. Math. Tables Aids Comput. **10** (1956), 82–86.

The authors consider a system of linear, ordinary differential equations of order n of the form

$$\frac{dY}{dt} = A(t)Y + F(t),$$

where $A(t)$ is a given square matrix, Y is a column matrix of the unknowns y_i, and the forcing functions form another column matrix $F(t)$. In a two-point boundary value problem r values of the components y_i are given in the initial point and $(n-r)$ values in the final point, and the paper is concerned with methods for converting such a problem into an initial value problem.

The first method utilizes the adjoint to the homogeneous equation and a one-dimensional form of Green's theorem; it requires $(n-r)$ integrations of the adjoint system and one of the originally given system. In the second method the homogeneous equation is used itself, and this procedure requires $(n-r)$ integrations of the homogeneous equation and r integrations of the nonhomogeneous equation.

Finally, the first approach is generalized to non-linear systems by introducing a certain iterative procedure.

P.-O. Löwdin (Uppsala).

Kliot-Dašinskiĭ, M. I. On a method of solution of a plane problem of potential theory. Leningrad. Inžen.-Stroit. Inst. Sb. Nauč. Trudov **17** (1954), 11–27. (Russian)

The author describes a method for the effective solution for the first boundary value problem for Poisson's equation in two-dimensional simply connected domains. The method uses complex orthogonal polynomials. The equation is written in the form $u_{z\bar{z}}=f$. One can find a function g such that $g_{\bar{z}}=f$. Then the function $h=u_z-g$ is analytic in the domain and may be expanded in a series of orthogonal polynomials. The coefficients can be expressed in terms of the given boundary value u_0. Once h is known u can be written down explicitly. The author estimates the error committed in using only a finite number of orhtogonal polynomials and computes a numerical example. The method is similar to the one of Rafalson [Dokl. Akad. Nauk SSSR (N.S.) **64** (1949), 799–802; MR **10**, 707]. *L. Bers* (New York, N.Y.).

Blanc, Charles. Sur l'intégration approchée d'équations du type parabolique. Z. Angew. Math. Phys. **7** (1956), 146–152.

This paper extends the author's previous work, on "stochastic error" study [cf. Comment. Math. Helv. **26** (1952), 225–241; Arch. Math. **5** (1954), 301–308; MR **14**, 691; **16**, 290], to examination of the errors resulting from one step in the solution, by each of three types of finite-difference formulae, to parabolic equations of the type $\partial u/\partial t - q\partial^2 u/\partial u^2 = f(x, t)$, with $u(x, 0)=g(x)$ given. A further paper is promised on the error effect of several steps in these processes. Conclusions regarding the relative merits of the formulae seem to be in agreement with those of other workers in this field. *L. Fox.*

See also: Hoffman and Kuhn, p. 370; Jaeckel, p. 370; Černikov, p. 371; Remez, p. 371; Neĭšuler, p. 383; Warmus, p. 391; Grebenyuk, p. 392; Malkin, p. 396; Vorob'ëv, p. 405; Kawasumi, p. 421; Robertson, p. 422; Wolff, p. 425; Shibacka, p. 435; Hurd, p. 442; McWeeny, p. 443; Sconzo, p. 447; Racer-Ivanova, p. 448; Andreoletti, p. 448; Renner, p. 449; Holsen, p. 449; Wolfe, p. 449; Mills, p. 450; Fei, p. 451; Prager, p. 451; Bellman, p. 451; Ward, p. 452.

Graphical Methods, Nomography

Sispánov, Sergio. Geometric solution of equations of third, fourth and fifth degree. Rev. Un. Mat. Argentina **17** (1955), 265–278 (1956). (Spanish)

In this paper is given a method of solution of a polynomial equation of third, fourth, or fifth degree with real coefficients. The method consists of graphing two lower degree equations and then of computation of the roots by algebra. *E. Frank* (Chicago, Ill.).

Manafu, Valeriu. La généralisation de la méthode des figures auxiliaires. Etude cinématique des groupes de 3-e classe. Bul. Inst. Politehn. Iaşi (N.S.) **2** (1956), 311–320. (Romanian. Russian and French summaries)

La détermination des vitesses v et des accélérations a dans certains mécanismes se peut faire en utilisant des points spéciaux, dont v et a se trouvent aisément. Dans les cas où ces points tombent en dehors de l'épure on utilise des figures auxilaires. Extension de cette méthode.

O. Bottema (Delft).

See also: Schwieger and Haberland, p. 432; Cox and Klein, p. 434; Koiter, p. 435.

Tables

★ **Table of square roots of complex numbers.** Report No. 10 of Numerical Computation Bureau, Tokyo, 1956. 21 pp.

The square root of the complex number $a+ib$ can quickly be made to depend upon the square root of $1+ix$ or $x+i$ with $0<x\leq 1$. Hence the table gives U, V, u, v, in

$$(1+ix)^{\frac{1}{2}} = U+iV,\ (x+i)^{\frac{1}{2}} = u+iv,$$

as functions of x. Eleven decimal place values are given,

with second differences, for $x=0(.002)1$. The table is a photolith of a very clear hand written copy.

D. H. Lehmer (Berkeley, Calif.).

★ **Jacobi, C. G. J. Canon arithmeticus. Nach Berechnungen von Wilhelm Patz, in verbesserter und erweiterter Form neu herausgegeben von Heinrich Brandt.** Akademie-Verlag, Berlin, 1956. 432 pp. DM 46.00.

This Canon is a completely reworked and greatly extended version of the now rare "Canon arithmeticus" [Typis Academicis, Berolini, 1839] of Jacobi. The actual author of the present work is W. Patz to whom many a number theorist will be grateful in the years to come. The tables of the Canon give for every modulus $m<1000$ possessing a primitive root g, a table of all powers of g and the inverse table of indices $I=\text{Ind}\, g\,(x)$ defined modulo $\phi(m)$ by

$$x\equiv g^{\text{Ind}(x)} \pmod{m}.$$

The 1839 Canon left out the moduli m of the form $2p^\alpha$ where p is an odd prime and $\alpha\geq 1$. There are thus 123 new moduli in the present table. A much more important extension however, is the inclusion of tables of

$$I'(\text{Ind}(x))=\text{Ind}(x+1)$$
$$I''(\text{Ind}(x))=\text{Ind}(x-1)$$

which play a role corresponding to addition and subtraction logarithms. Thus

$$\text{Ind}(a+b)=\text{Ind}(b)+\text{Ind}\left(\frac{a}{b}+1\right),$$

$$\text{Ind}(a-b)=\text{Ind}(b)+\text{Ind}\left(\frac{a}{b}-1\right),$$

so that one may enter the I' table with argument $\text{Ind}\, a-\text{Ind}\, b$ and add $\text{Ind}(b)$ to obtain the index of $a+b$ without knowing a or b. Thus polynomials may be evaluated mod m without too much difficulty. Problems like finding three consecutive quintic residues modulo 481 are solved on sight. The 16 page introduction by Brandt gives 24 different examples of the use of the table.

The choice of primitive root in the present canon is the least positive one. This choice was rarely made by Jacobi. In fact in only 12 tables do the two editions agree. The choice of least primitive root would seem to be a good one for the user although Patz paid something in new computing for not adopting some simpler primitive roots like ± 10 or its fractional powers the way Jacobi did.

The revival of the Canon Arithmeticus in its new improved form will do much to aid the exploration of many problems dealing with moduli less than 100 especially for users with only hard computing facilities.

In the reviewer's copy pages 50, 51, 62, 63 are not printed. It is hoped that this omission is unique.

D. H. Lehmer (Berkeley, Calif.).

Krarup, Torben; and Svejgaard, Bjarner. Geodetic Tables: International Ellipsoid. Calculated under the direction of Einar Andersen. Geodaet. Inst. Skr. (3) **24** (1956), 8+181 pp.

These tables present seven geodetic latitude (φ) functions on the International (Hayford) ellipsoid for $0\leq\varphi\leq 90°$ in one minute of arc intervals. The functions tabulated are: (1) $W=(1-e^2\sin^2\varphi)^{\frac{1}{2}}$, where e is the eccentricity of the ellipsoid, to 12 significant figures (s.f.); (2) $N^{-1}(10^7)$, where N is the radius of curvature in the prime vertical, to 12 s.f.; (3) $M^{-1}(10^7)$, where M is the radius of curvature in the meridian, to 12 s.f.; (4) γ, the theoretical force of gravity on the ellipsoid, to .01 milligal; (5) $(\varphi-\varphi^*)$, where

$$\tan\left(\tfrac{1}{2}\varphi^*+\tfrac{1}{4}\pi\right)=\tan\left(\tfrac{1}{2}\varphi+\tfrac{1}{4}\pi\right)(1-e\sin\varphi)^{\frac{1}{2}e}(1+e\sin\varphi)^{-\frac{1}{2}e}$$

to 10^{-6} seconds of arc; (6) $(\beta-\varphi)$, where $\tan\beta=(1-e^2)^{\frac{1}{2}}\tan\varphi$, to 10^{-6} seconds of arc; (7) $\rho/2MN$, where ρ is the number of seconds in a radian, to 6 s.f.

The functions were calculated directly from trigonometrical series expansions in $\sin 2\lambda\varphi$ or $\cos 2\lambda\varphi$ $(0\leq\lambda\leq 4)$ (except for (4) where $0\leq\lambda\leq 3$) for every degree of φ. Difference formulas to the fifth order were drawn up and used to compute the functions to the nearest minute.

The same, or similar, functions have appeared elsewhere, but not to as many s.f. as listed here. In order to retain the same number of s.f. when interpolating in the tables, it is necessary to use quadratic interpolation in (1), (2), (3), (5) and (6).

The introduction to the tables is concise but thorough, and gives complete reference to the derivation of the formulas.

B. Chovitz (Washington, D.C.).

Kawasumi, Hirosi. Notes on the theory of vibration analyser. Bull. Earthquake Res. Inst. Tokyo **34** (1956), 1–8. (Japanese summary)

Author presents the theory of an analogue computer used for the harmonic analysis of the spectrum of seismic waves. Line- and continuous spectra are analysed by linear oscillators with and without damping respectively.

W. Freiberger (Providence, R.I.).

★ **Tables of Whittaker functions (wave functions in Coulomb field).** Report No. 9 of Numerical Computation Bureau. The Tsuneta Yano Memorial Society, Tokyo, 1956. 67 pp.

This is the second volume devoted to Whittaker functions to be produced by the Tokyo Numerical Computation Bureau [see same Rep. No. 8 (1954); MR **16**, 175] and concerns the function

$$W=W_{i\xi,\frac{1}{2}}(-ix)$$

or rather the modified function

$$\exp\{-\tfrac{1}{2}x\xi+i\sigma(\xi)\}W=G_\xi(x)+iF_\xi(x),$$

where $\sigma(\xi)=\arg\Gamma(1-i\xi)$. For $0<x<1$ it is more convenient to tabulate the function $\Gamma_\xi(x)$ defined by

$$[F_\xi'(0)]^2\{\Gamma_\xi(x)-G_\xi(x)\}=\xi\log xF_\xi(x)$$

instead of G. Modifications of the absolute value and argument of W and its derivative with respect to x are defined as follows:

$$U_\xi(x)=|G+iF|,\quad V_\xi(x)=2|F'-iG'|,$$
$$\phi_\xi(x)=\arg(F'-iG')-\omega_\xi(x),$$
$$\psi_\xi(x)=\arg(F'-iG')-\omega_\xi(x),$$

where $\omega_\xi(x)=\sigma(\xi)+\frac{1}{2}x+\xi\log x$. There are three tables all to six decimals with second central differences in both and ξ directions. Table 1 gives F, F', Γ, Γ' for $\xi=0(.02).5(.05)1$, and $x=0(.05)1$. Table 2 gives F, F', G, G' for $\xi=0(.02).5(.05)1$ and $x=1(.05)2(.1)5$. Table 3 gives U, V, ϕ, ψ for $\xi=.1(.1)1$, $x^{-1}=0(.05).2$. There is a misprint of ξ for y on p. 5 which is misleading. The tables are beautifully printed and should be quite useful.

D. H. Lehmer.

Fieller, E. C.; Lewis, T.; and Pearson, E. S. Corrigenda to: Correlated random normal deviates. Tracts for computers, no. 26. Biometrika **43** (1956), 496–497.

The photo-offset method used experimentally in preparing these tables [Cambridge, 1955; MR **17**, 638] did not allow sufficient checking in proof. Errors are relevant in this table of "random" numbers since check sums are given. Also the errors mainly consisted of losing minus signs and hence "biased" the table. *I. R. Savage.*

See also: Blagoveščenskiĭ and Fil'čakov, p. 399; Salzer, p. 416; Läuchli, p. 423; Chu, p. 423; Wolff, p. 425; Fox, p. 426; Romani, p. 427.

Machines and Modelling

★ **Wilkes, M. V. Automatic digital computers.** John Wiley and Sons, Inc., New York, 1956. 305 pp. $7.00.

Many mathematicians who are concerned with computers are content to accept their behavior axiomatically. For those who are curious as to the realities behind the console, the present volume is a pleasant guide. It is largely self-contained, granted a familiarity e.g. with the basic behavior of vacuum tubes — a summary of which might have been useful to those who have been long away from an elementary physics laboratory. The treatment is qualitative: do-it-yourself building of computers is explicitly discouraged in the preface and implicitly by the fact that only in one circuit diagram are tube types and circuit parameters given.

The first chapter is mainly historical. The second is concerned with the principles of logical design, exemplified by a discussion of ESDAC. The third chapter deals with the principles of program construction, including the use of the computer itself for this purpose. The fourth chapter discusses relay computers. Chapter five is concerned with storage systems of various kinds. In the sixth chapter algebraic representations of switching circuits are considered. The last chapter discusses, in general terms, the design and operation of digital computers. There is an appendix discussing "Machinery and intelligence" and a bibliography. *J. Todd.*

Stock, John Robert. Die mathematischen Grundlagen für die Organisation der elektronischen Rechenmaschine der Eidgenössischen Technischen Hochschule. Mitt. Inst. Angew. Math. Zürich no. 6 (1956), 73 pp.

This gives an account of the logical design of the ERMETH. This is a one-address serial floating decimal machine, with a magnetic drum memory of 10,000 words of 16 digits. A floating-decimal number is of the form $a\cdot 10^b$, where $-200\leq b\leq 199$ and $|a|<10$; alternatively a word can represent a fixed decimal number with 14 decimal digits. There is another decimal digit in the word, which serves two purposes, as a parity check (mod 3) and as an aid (Q-sign) for a programming system devised by Rutishauser [same Mitt. no. 3 (1952); MR **15**, 64]. An instruction consists of 7 d.d.: four for the address, two for the operation and a third indicating one of nine index registers. The machine arithmetically is roughly a 10 ms. one and has an average access time of about 5 ms. The normal input is by means of punched cards.

The first introductory chapter includes a list of the instructions. The second describes the actual form of the operations. The third chapter discusses the principles of realization of the machine. The fourth chapter is concerned with the logical design of the various components. There are appendices which give block diagrams of the arithmetic unit and the control. There are a few simple examples of programming for the machine which illustrate the use of index register and the method of reading in a program; there is also a sub-routine for square root and a program for the solution of a system of equations by an elimination process, which uses the index-register and the Q-sign. *J. Todd.*

Bauer, Walter F. Modern large scale computer system design. Computers and Automation **6** (1957), no. 1, part 1, 8–25, 32, 34.

Robertson, H. H. Phase calculations for nuclear scattering on the Pilot ACE. Proc. Cambridge Philos. Soc. **52** (1956), 538–545.

A particular high-speed automatic computing machine is used for solving integro-differential equations of the form

$$\frac{d^2f}{dr^2}+u(r)f(r)=\int_0^R K(r,r')f(r')dr',$$

where R is large, $u(r)=k^2-l(l+1)r^{-2}+V(r)$, and K and V vary with the reaction considered. The method, not peculiar to one machine, uses finite-difference formulae for the integral and second derivative, and so replaces the integrodifferential equation by a set of algebric equations, whose unknowns are the required values of $f(r)$ at equidistant points in the range $0\leq r\leq R$. The phase is determined by comparing two numerical values with the known form of the analytical solution: There is also a description of methods for evaluating the kernel in certain representative cases, a note on previous methods, and one numerical example. *L. Fox.*

Routledge, N. A. Logic on electronic computers: a practical method for reducing expressions to conjunctive normal form. Proc. Cambridge Philos. Soc. **52** (1956), 161–173.

A proposition letter formula in conjunctive normal form is tautologous if and only if each conjunct is tautologous, i.e., contains the disjunction of a letter with its negation. By means of an ingenious algorithm especially suited for binary notation (and computers) any formula containing as connectives only $\vee$, &, $\sim$, with the latter restricted to single letters (here called a simple formula) is reduced to conjunctive normal form. The procedure is analyzed syntactically using parenthesis-free notation and is essentially a systematic way of recording all the distinct conjuncts obtainable by "adding out" using the distributive law. Each disjunct is tested as formed, and as soon as one is found that fails, the formula is rejected. A procedure is also described for reducing any formula in only the above connectives to a simple formula. The algorithms are then combined, taking advantage of several short-cuts. The algorithm was programmed for the Ace Pilot model at the National Physical Laboratory for formulas with a maximum length of 160 symbols and 30 distinct variables. No information is given concerning the time required to carry out the test. Erratum: page 163, line 14, for "an" read "the first." *G. W. Patterson.*

Reed, Harry L., Jr. Machine computations for nonlinear exterior ballistics. Ordnance Computer Research Report, Ballistic Research Laboratories, Aberdeen Proving Ground, Md. vol. 3 (1956), no. 3, pp. 3–5. (Government Agencies, their contractors and others cooperating in Government research may obtain reports directly from the Ballistic Research Laboratories. All others may purchase photographic copies from the Office of Technical Services, Department of Commerce, Washington 25, D.C.)

This paper is a short essay on the methods of solutions used to handle the problem on computing machines.

R. W. Hamming (Murray Hill, N.J.).

Schecher, Heinz. Massnahmen zur Vereinfachung von Rechenplänen bei elektronischen Rechenanlagen. Z. Angew. Math. Mech. **36** (1956), 377–395. (English, French and Russian summaries)

The paper deals with programming techniques for a machine possessing index registers and a provision for making automatic substitution of addresses.

A. S. Householder (Madison, Wis.).

Läuchli, P. Berechnung und Drucken einer achtstelligen Logarithmentafel als Beispiel für das Arbeiten eines Rechenautomaten. Elem. Math. **11** (1956), 130–134.

Yanowitch, M. The solution of boundary value problems on a REAC analog computer. Computers and Automation **6** (1957), no. 2, 26–29, 39.

See also: McNaughton, p. 370; Young, p. 417; Bondar', p. 428; Dmitriev, p. 448.

PROBABILITY

Narayana, T. V. A problem in the theory of probability. J. Indian Soc. Agric. Statist. **6** (1954), 139–146.

The author discusses, using generating functions, a coin tossing game which ends the first time the total number of heads exceeds the total number of tails by exactly r. [See Feller, An introduction to probability theory and its applications, v. 1, Wiley, New York, 1950; MR **12**, 424].

J. L. Doob (Geneva).

Ludwig, Otto. Die Pascalsche Fragestellung für Merkmalsiterationen (runs). Z. Angew. Math. Mech. **36** (1956), 264–265.

Morgenstern, Dietrich. Einfache Beispiele zweidimensionaler Verteilungen. Mitteilungsbl. Math. Statist. **8** (1956), 234–235.

Geisser, Seymour. The modified mean square successive difference and related statistics. Ann. Math. Statist. **27** (1956), 819–824.

The modified mean square successive difference δ_0^2 is defined for a sequence $X_1, \cdots, X_{2m}$ by

$$4(m-1)\delta_0^2=\sum_1^{m-1}(X_{i+1}-X_i)^2+\sum_{m+1}^{2m-1}(X_{i+1}-X_i)^2.$$

When the X_i are independent $N(0, \sigma^2)$ the p.d. of δ_0^2/σ^2 is expressible as an alternating mixture of scaled exponentials [A. R. Kamat, Sankhyā **15** (1955), 295–302; MR **17**, 503]. Here this expression is used to obtain the moments of δ_0^2, the p.d. of substitute F and t ratios, and the .025 and .075 quantiles of δ_0^2/σ^2 and substitute t for $2m=4(2)50$.

J. Hannan (Stanford, Calif.).

Dall'Aglio, Giorgio. Sugli estremi dei momenti delle funzioni di ripartizione doppia. Ann. Scuola Norm. Sup. Pisa (3) **10** (1956), 35–74.

Let Φ, Ψ be specified one-dimensional distribution functions and let m, p, α be arbitrary numbers except that $\alpha\geqq 1$. The author considers the integral

$$\iint_{-\infty}^{\infty}|y-mx-p|^\alpha d\Phi(x, y),$$

where $\Phi(x, y)$ defines a distribution function with marginal distribution functions Φ, Ψ. Appropriate hypotheses are imposed to insure the existence of the indicated moment. If $\alpha=1=m$, and if $p=0$, the integral has the minimum value $\int_{-\infty}^{\infty}|\Phi(x)-\Psi(x)|dx$. There is a unique minimizing $\Phi(x, y)$ or else infinitely many such minimizing functions. Corresponding results are proved for the maximum value of the integral, and these results are generalized to cover arbitrary values of m, p. When $\alpha>1$, however, the situation changes. There is a unique minimizing and a unique maximizing $\Phi(x, y)$, both of which are found explicitly.

J. L. Doob (Geneva).

Richter, H. Zur Abschätzung von Erwartungswerten. Z. Angew. Math. Mech. **36** (1956), 266.

Chu, J. T. Errors in normal approximations to the t, τ, and similar types of distribution. Ann. Math. Statist. **27** (1956), 780–789.

It is known that several sequences of distributions tend rapidly to a normal distribution but very little is known about the bounds of error made in using the normal approximation. By making use of some simple inequalities and deriving a set of other inequalities involving the cumulative distribution function (cdf) of a standard normal variate, the author has obtained upper and lower bounds of error in using the normal approximation for several cdf's. In particular he has shown that the proportional errors are smaller than $1/n$ for all $n\geqq 8$ in the case of Student's t distribution and for all $n\geqq 13$ in the case of Thompson's τ distribution, where n is the number of degrees of freedom. The author has also obtained bounds on the error in using normal approximation for the distribution of the partial and total correlation coefficients when the variates involved are independently and normally distributed. He states that in the case of χ^2-distribution, although he has obtained some simple results, he was unsuccessful in deriving both upper and lower bounds similar to those obtained in the obtained in the case of t and τ distributions. Tables are given showing the results of some numerical comparisons of two known approximations for t distribution and the upper and lower bounds of error for each of these approximations.

Om P. Aggarwal (Edmonton, Alta.).

Makabe, Hajime; and Morimura, Hidenori. On the approximation to some limiting distributions. Kōdai Math. Sem. Rep. **8** (1956), 31–40.

The author obtains a new and useful approximation to the Poisson distribution (individual and cumulative

probabilities) in terms of the binomial distribution, with an upper limit of the error. He also finds an upper limit for the discrepancy between the normal distribution $N(0.1)$ and the distribution of the properly normed sum of n independent random variables, each uniformly distributed on the interval (0.1). The latter is an improvement on an error estimate given by Uspensky [Introduction to mathematical probability, McGraw-Hill, New York, 1937]. *J. L. Doob* (Geneva).

★ **Obreschkoff, Nikola. Über einige asymptotische Formeln in der Wahrscheinlichkeitsrechnung.** Bericht über die Tagung Wahrscheinlichkeitsrechnung und mathematische Statistik in Berlin, Oktober, 1954, pp. 37–42. Deutscher Verlag der Wissenschaften, Berlin, 1956.

The author obtains several asymptotic formulas for the cumulative distribution function of the sum $X_1+\cdots+X_n$, where the random variables $X_1, \cdots, X_n$ are independent and identically distributed. Separate approximations are given for the continuous and discrete cases. These asymptotic formulas are then applied to the Bernoulli and Poisson distributions. [See also Obrechoff, Annuaire [Godišnik] Univ. Sofia. Fac. Sci. Livre 1. (Math. Phys.) 44 (1948), 201-233; MR 12, 269.]
H. P. Edmundson (Santa Monica, Calif.).

Austin, Donald G. Some differentiation properties of Markoff transition probability functions. Proc. Amer. Math. Soc. 7 (1956), 751–761.

Proofs and further details of results announced in a previous paper [Proc. Nat. Acad. Sci. U.S.A. 41 (1955), 224–226; MR 16, 1130]. *J. L. Doob* (Geneva).

Wang, Shou-Jen. On the differentiabilities of the transition probabilities in temporally-homogeneous Markoff process with enumerable number of states. Acta Math. Sinica 4 (1954), 359–364. (Chinese. English summary)

The proof is incorrect. The inequality on p. 362 that

$$p_{ij}(t_0+t)\geqq \sum_{k=1}^{n} Q_{k-1}p_{ij}(t_0+h)p_{ij}(t-kh)$$

does not necessarily hold if $t_0>0$ since the corresponding events are not mutually exclusive as in the Kolmogorov case $t_0=0$. *K. L. Chung* (Chicago, Ill.).

Dobrušin, R. L. On the condition of the central limit theorem for inhomogeneous Markov chains. Dokl. Akad. Nauk SSSR (N.S.) 108 (1956), 1004–1006. (Russian)

If $P(x, B)$ is a transition probability, the author defines

$$\beta=1-\sup_{x,B}|P(x, B)-P(y, B)|.$$

Then $\beta\geqq\rho$, where ρ is the coefficient of ergodicity defined in a previous paper [same Dokl. (N.S.) 102 (1955), 5–8; MR 17, 48]. Several theorems on the asymptotic normality of sums of random variables in a Markov process given in that paper remain true if ρ is replaced by β. The following are among further theorems stated. Let β_n be the minimum β value of the transition functions involved in the sum ζ_n of n random variables $\{\xi_{in}, i\leqq n\}$ of a Markov process. (I) If

$$\lim_{n\to\infty}(n\beta_n^2)^{-1}\sum_{i=1}^{n}\int_{|t|\geqq\epsilon n^{1/2}\beta_n^{3/2}} t^2dP\{\xi_{in}-M\xi_{in}<t\}=0$$

for every $\epsilon>0$, and if $0<\text{const}\leqq\text{Var}\,\xi_{in}\leqq\text{const}<\infty$, then ζ_n is asymptotically normally distributed (with the usual centering and scaling constants). (II) If $|\xi_{in}|\leqq\text{const}<\infty$, and if

$$\lim_{n\to\infty} n^{-2/3}\beta_n\Big(\sum_{i=1}^{n}\text{Var}\,\xi_{in}\Big)=\infty,$$

then ζ_n is asymptotically normally distributed.
J. L. Doob (Geneva).

Takács, L. On a probability problem arising in the theory of counters. Proc. Cambridge Philos. Soc. 52 (1956), 488–498.

This work was stimulated by a paper of J. M. Hammersley [same Proc. 49 (1953), 623–637; MR 15, 139]. The author uses however a different mathematical model for the same counter problem since he does not substitute a circumference for the time interval. Nevertheless his asymptotic solution for the type II counter agrees exactly with the solution given in the first half of Hammersley's paper. In the last section the author investigates a process in which the sequence of arrival times is subject to two consecutive transformations. First the arrival times are registered by a type II counter and the resulting counts are then subjected to a transformation by a type I (or type II) counter. Due to the difference in models the results obtained do not agree with Hammersley's solution for the problem of paralysis. The author uses the method which he employed in a series of papers [see, e.g., Acta Math. Acad. Sci. Hungar. 6 (1955), 101–129; MR 17, 51]; the article quoted contains already some of the results of the present paper. *E. Lukacs.*

Patlak, Clifford S. Random walk with persistence and external bias. Bull. Math. Biophys. 15, (1953) 311–338.

The author obtains the partial differential equation of a random walk with persistence of direction and external bias. By persistence of direction or internal bias is meant that the probability a particle travels in a given direction is not necessarily the same for all directions, but depends only on the particle's previous direction of motion. By external bias is meant that the probability a particle travels in a given direction is dependent upon an external force on the particle. For the net displacement of a particle it is assumed that (1) the probability of travel in any direction after turning and (2) the distance of travel in a given direction are not necessarily the same for all directions. A modified Fokker-Planck equation is derived under the assumptions that the particles have a travel time and speed distribution and that the expected travel time between turns need not be zero. From this the desired partial differential equation for the random walk with persistence and external bias is obtained. Applications to physics and chemistry are given by diffusion and long-chain polymer examples. The introduction summarizes results on random walks up to 1952 and the list of references is very comprehensive. *H. P. Edmundson.*

Chung, K. L.; and Derman, C. Non-recurrent random walks. Pacific J. Math. 6 (1956), 441–447.

Let $X_1, X_2, \cdots$ be independent versions of a random variable X with integer values and $E(X, X\leqq 0)>-\infty$, such that the greatest common divisor of

$$(m: \Pr(X=m)>0)$$

is 1; let $S_n=X_1+X_2+\cdots+X_n$ for $n\geqq 1$; and let i.o. stand for infinitely often. Chung and Fuchs [Mem.

Amer. Math. Soc., no. **6** (1951); MR **12**, 722] have shown that, for each integer m, $\Pr(S_n=m, \text{i.o.})=1$ or 0 according as $E(X)=0$ or not. Elaborating the second alternative, the authors show (1) that if $0<E(X)<+\infty$ and if I contains an infinite number of non-negative integers, then $\Pr(S_n \in I, \text{i.o.})=1$, (2) that if $E(X)=+\infty$, then $\Pr(S_n \in I, \text{i.o.})$ is sometimes 0, and (3) that (1) and (2) continue in force if $\Pr(X \leqq \alpha)$ is smooth enough.

H. P. McKean, Jr. (Princeton, N.J.).

Wolff, Hans-Dieter. Die Reserve-Berechnung linear steigender Versicherungssummen. Bl. Deutsch. Ges. Versicherungsmath. **3** (1956), 97–101.

The author proposes an approximation formula for the premium reserves of a temporary assurance for linearly increasing amounts. The approximation is based on empirical curve fitting. An appended table compares exact and approximated values and indicates that the percentage error is large. The method can nevertheless be useful since these assurances occur invariably only in connection with some other combination stipulating the return of paid premiums. In this case the reserve for the linearly increasing assurance is only an insignificant portion of the total reserve. *E. Lukacs.*

Wolff, Karl-H. Das Theorem von de Finetti für mehrere Ausscheideursachen. Statist. Vierteljschr. **9** (1956), 70–79.

Schmetterer, Leopold. Die Risikotheorie in der Versicherungsmathematik. II. Statist. Vierteljschr. **9** (1956), 47–63.

Leepin, Peter. Über den Einfluss von Änderungen der Rechnungsgrundlagen auf Prämien und Prämienreserven. Bl. Deutsch. Ges. Versicherungsmath. **3** (1956), 3–22.

This article gives an expository treatment of some of the variational problems of actuarial mathematics. The author's emphasis on an elementary presentation should help to familiarize the practising actuary with the available techniques. *E. Lukacs* (Washington D.C.).

See also: Špacek, p. 384; Obuhov, p. 388; Diederich, p. 439; Chintschin, p. 443; Blackwell, p. 450; Fennell and Oshiro, p. 451; Macey, p. 452.

STATISTICS

Brandner, Fred A. Common elements. Proc. Iowa Acad. Sci. **63** (1956), 528–533.

If X and Y are affected by s equally likely causes of which t are common to both, their correlation is obtained by a new definition, consistent with earlier ones but more widely applicable.

Karlin, Samuel; and Rubin, Herman. The theory of decision procedures for distributions with monotone likelihood ratio. Ann. Math. Statist. **27** (1956), 272–299.

The essential completeness of a class of monotone decision procedures for certain k-decision problems for distributions of exponential type was proved, using the monotone nature of the likelihood ratio, by Sobel in his thesis [Columbia Univ., 1951]. The monotone likelihood ratio was exploited more generally by Rubin in an unpublished paper which the present authors mention as having many points in common with the present paper and which is a reference for Section 7.4 of the book "Theory of games and statistical decisions" [Wiley, New York, 1954, hereafter denoted BG; MR **16**, 1135] by Blackwell and Girshick which treats the exponential class with finitely many decisions. Other recent papers deal with similar topics. The conclusions of the present paper for families with the monotone likelihood ratio property (see p. 187 of BG) are similar to those of BG in the case of the exponential class. Thus, the authors characterize the Bayes strategies as monotone and the monotone strategies as admissible or essentially complete or Bayes with respect to a priori distributions concentrated on certain finite sets, etc., under various sets of conditions too lengthy to detail here. *J. Kiefer.*

Konijn, H. S. Some estimates which minimize the least upper bound of a probability together with the cost of observation. Ann. Inst. Statist. Math., Tokyo **7** (1956), 143–158.

Nonsequential results for estimating the mean of a normal distribution with known variance, using the method of Wolfowitz [Ann. Math. Statist. **21** (1950), 218–230; MR **12**, 36], for some slightly different weight functions than those considered by the latter.

J. Kiefer (Ithaca, N.Y.).

★ **Elfving, Erik Gustav. Über optimale Allokation.** Bericht über die Tagung Wahrscheinlichkeitsrechnung und mathematische Statistik in Berlin, Oktober, 1954, pp. 89–95. Deutscher Verlag der Wissenschaften, Berlin, 1956.

This is in effect an expository paper, amplifying on the author's results in Skand. Aktuarietidskr. **37** (1954), 170–190 [MR **17**, 640]. *J. Isbell* (Princeton, N.J.).

Rao, C. Radhakrishna; and Chakravarti, I. M. Some small sample tests of significance for a Poisson distribution. Biometrics **12** (1956), 264–282.

A variety of small-sample tests connected with the Poisson distribution are presented; they are based on the principle that for such a distribution, the conditional distribution of n observations given their sum is independent of the parameter. (1) To test that f observations come from the same Poisson distribution, two tests are proposed, the Chi-square test and the likelihood ratio test. Some moments of the distribution of the former are found. (2) To test whether or not the frequency of the zero observation is compatible with the hypothesis that the underlying distribution is Poisson, the Chi-square and likelihood ratio tests are briefly considered; the exact test is reduced to a classical occupancy problem. (3) Similar tests are considered for the truncated Poisson, where, e.g., zero observations are not recorded. *K. J. Arrow.*

Masuyama, Motosaburo. On a fundamental formula in bulk sampling from the viewpoint of integral geometry. Rep. Statist. Appl. Res. Un. Jap. Sci. Engrs. **4** (1956), 85–89.

The author notes that the sampling variance of the estimate of a double or triple integral, based on observations at sample points, is diminished, in general, if instead of observing values of the integrand itself we observe values of a weighted local average of it. He had

noted a special case of the formulae concerned previously [same Rep. 2, (1953) no. 4, 113–119, § 3; MR 15, 332].
H. P. Mulholland (Birmingham).

Anscombe, F. J. On estimating binomial response relations. Biometrika 43 (1956), 461–464.

The author reviews the problem of fitting a logistic form of dose-response law by maximum likelihood and shows that it is simpler than fitting an integrated normal law by the same method. He also examines Berkson's method of fitting the logistic dose response law [essentially by modified minimum χ^2 procedures] and suggests a small modification of it. *D. G. Chapman.*

Gilbert, N. E. G. Likelihood function for capture-recapture samples. Biometrika 43 (1956), 488–489.

Since samples taken in the capture-recapture method of estimating the size of a wild animal population are usually without replacement, the finite sampling correction should be used. The reviewer has suggested a method of doing this [Univ. California Publ. Statist. 1 (1951), 131–159; MR 13, 52]. The present note suggests a modification of this, though no discussion is given whether this is better in some sense. One numerical example is given. *D. G. Chapman* (Seattle, Wash.).

Creasy, Monica A. Confidence limits for the gradient in the linear functional relationship. J. Roy. Statist. Soc. Ser. B. 18 (1956), 65–69.

Let x_i, y_i denote a sample, $(i=1, 2, \cdots, n)$, drawn from n bivariate normal populations with means ξ_i, η_i, common variances δ_x^2, δ_y^2, and zero correlations; also let $\eta_i=\beta+\alpha\xi_i$. The point estimation of the parameters has been considered by Lindley [Suppl. J. Roy. Statist. Soc. 9 (1947), 218–244; MR 9, 363]. The author finds confidence intervals for α for both normally distributed and fixed ξ_i, η_i, distinguishing the cases $\beta\neq0$ and $\beta=0$. She generally assumes that the ratio δ_x^2/δ_y^2 is known and that $\Pr\{|\arctan\hat{\alpha}-\arctan\alpha|>\frac{1}{2}\pi\}$ is negligibly small.
H. A. David (Melbourne).

Blank, A. A. Existence and uniqueness of a uniformly most powerful randomized unbiased test for the binomial. Biometrika 43 (1956), 465–466.

Let θ denote the probability of a success in a binomial trial. Tocher [Biometrika 37 (1950), 130–144; MR 12, 193] has shown that if a uniformly most powerful test among unbiased tests of the hypothesis $\theta=p$ against the alternative $\theta\neq p$ exists, it must be of the form $\psi_k=1$, $s+1\leq k\leq t-1$; $\psi_s=c$; $\psi_t=d$; $\psi_k=0$ otherwise, where ψ_k, $0\leq k\leq n$, is the probability that the null hypothesis is accepted if the observed number of successes is k. The present paper shows that among tests of this type there exists a unique unbiased test. *G. E. Noether.*

James, G. S. On the accuracy of weighted means and ratios. Biometrika 43 (1956), 304–321.

Let X_i, s_i^2, $1\leq i\leq k$, be independent, X_i being normal with mean μ and variance $\lambda_i\sigma_i^2$, and s_i^2/σ_i^2 having a χ^2 distribution with ν_i degrees of freedom. Only the λ_i and ν_i are known. Let $r_i=(\lambda_i s_i^2)^{-1}/\sum_i(\lambda_i s_i^2)^{-1}$. The main result of the paper is to obtain a function $u(r_1, \cdots, r_K)$ such that the confidence interval

$$\sum r_iX_i\pm u(r_1, \cdots, r_K)/[\sum_i(\lambda_i s_i^2)^{-1}]^{1/2}$$

on μ has constant confidence coefficient to within terms of order ν_i^{-4} (the classical procedure gives ν_i^{-1} here). The function u is tabulated when $k=2$. The paper is related to previous work of the author [Biometrika 41 (1954), 19–43; MR 16, 842] and of Welch [ibid. 34 (1947), 28–35; MR 8, 394]. *J. Kiefer* (Ithaca, N.Y.).

Severo, Norman C.; and Olds, Edwin G. A comparison of tests on the mean of a logarithmico-normal distribution with known variance. Ann. Math. Statist. 27 (1956), 670–686.

If X is ln-normal with variance γ^2 and mean $\theta\gamma$, $Y=\ln(X/\gamma)$ is normal with variance $\sigma^2=\ln 1+\theta^{-2}$ and mean $\mu=\ln\theta-\sigma^2/2$. The tests of $\theta=\theta_0$ whose OC (Operating Characteristic) functions are compared for $\theta>\theta_0$ are specified by critical regions of the types T_1: $\sum Y_i>n\mu_0+n^{\frac{1}{2}}\sigma_0Z_{1-\alpha}$, T_2: $\sum e^{Y_i}>n\theta_0+n^{\frac{1}{2}}Z_{1-\alpha}$, and, the Neyman-Pearson best against $\theta_1>\theta_0$, $T_3(\theta_1)$:

$$\sum(Y_i-k)^2<\sigma_0^2\chi_\alpha^2(n;\, n(\mu_0-k)^2/\sigma_0^2)$$

with $k=\mu_0+(\mu_1-\mu_0)(1-\sigma_1^2/\sigma_0^2)^{-1}\uparrow$ with respect to θ_1. {Not noted is that csf $T_3(\theta_1)\to$csf T_1 as $\theta_1\to\infty$ and $\to$ the locally most powerful test as $\theta_1\to\theta_0$.} OC-functions are computed for $\alpha=.05$, $n=4$, 25, $\theta_0=1$, 10, $\theta_1-\theta_0=1$, 9. It is proved that $\mathrm{OC}(\theta_0+\delta|T_i)\to\Phi(Z_{1-\alpha}-n^{\frac{1}{2}}\delta)$ as $\theta_0\to\infty$ for fixed $n, \delta, i, \theta_1-\theta_0>0$. *J. Hannan* (Stanford, Calif.).

Cox, D. R. A note on the theory of quick tests. Biometrika 43 (1956), 478–480.

Let t and q be unbiased estimates of a parameter θ, with efficiencies 1 and E, respectively, and let t' and q' be the normal deviates associated with the corresponding tests of the hypothesis $\theta=\theta_0$, using asymptotic sampling variances. Then, for given q' and assuming a uniform prior distribution for θ, t' has mean $q'E^{-\frac{1}{2}}$ and variance $(1-E)/E$. Thus the result of the efficient test may be predicted, within limits, from that of the inefficient test.
P. Armitage (London).

Moran, P. A. P. A test of significance for an unidentifiable relation. J. Roy. Statist. Soc. Ser. B. 18 (1956), 61–64.

The hypothesis tested is that the line corresponding to an unidentifiable linear relation passes through a specified point. [For the problem of identification of linear relations between variables subject to error see, e.g., Reiersøl, Econometrica 18 (1950), 375–389; MR 12, 347; and Jeeves, Ann. Math. Statist. 25 (1954), 714–723; MR 16, 604.] *D. M. Sandelius* (Göteborg).

Fox, Martin. Charts of the power of the F-test. Ann. Math. Statist. 27 (1956), 484–497.

This paper presents charts of the power of the F test designed to simplify entry and interpolation. Charts are presented for significance level $\alpha=.05$ and .01, power $\beta=.5$, .7, .8 and .9. *H. Chernoff* (Stanford, Calif.).

Ruben, H. On the sum of squares of normal scores. Biometrika 43 (1956), 456–458.

Let $x_{r|n}$ $(r=1, 2, \cdots, n)$ be the rth order statistic in a sample of n from a standardised normal population, and $S_n=\sum_{r=1}^n a_{r|n}^2$, where $a_{r|n}=E(x_{r|n})$. S_n is expressed in terms of the contents of regular hyperspherical simplices with primary angles $\cos^{-1}(-\frac{1}{2})$. *P. Armitage.*

Moriguti, Sigeiti. Efficiency of a sampling inspection plan. Rep. Statist. Appl. Res. Un. Jap. Sci. Engrs. 4 (1956), 71–77.

Under suitable regularity conditions, the author

proves that for a sequential test on independent random variables with common density f_θ, the average sample number satisfies $E_\theta n \geq [L'(\theta)]^2/I_\theta L(\theta)[1-L(\theta)]$, where L is the OC function and $I_\theta = -E_\theta \partial^2 \log f_\theta(X_1)/\partial\theta^2$. Examples are given where the ratio of this lower bound to $E_\theta n$ is computed. {Reviewer's comment: the inequality is a special case of that of Wolfowitz [Ann. Math. Statist. **18** (1947), 215–230; MR **9**, 49], obtained by letting the "estimator" be equal to the characteristic function of the acceptance region.} *J. Kiefer* (Ithaca, N.Y.).

Romaní, José. Non-parametric tests in sequential form. Trabajos Estadist. **7** (1956), 43–96. (Spanish. English summary)

The author considers several non-parametric hypotheses and reduces them to parametric problems by neglecting information, as is done in many of the standard procedures. He then constructs sequential parametric tests for these problems. The tests are constructed in detail and the Wald approximation is used throughout to obtain the test procedures and the OC and ASN curves. Tables are given of some of the tests and their properties.

The problems considered are (1) one-sided test of median; (2) the test that two samples of the same size come from the same population with a one-sided slippage alternative; (3) the same with a two-sided slippage alternative; (4) tests of equality of "dispersion"; (5) combined test of slippage and dispersion; (6) general test of equality of populations; (7) test of randomness. Problem (1) is analysed in detail, and the author computes the efficiency of his test against the normal sequential test (about 60%) and against the normal non-sequential test (120%). *H. Rubin* (Eugene, Ore.).

Albert, G. E. Accurate sequential tests on the mean of an exponential distribution. Ann. Math. Statist. **27** (1956), 460–470.

"In this paper, methods introduced by the author [same Ann. **25** (1954), 340–356; MR **15**, 973] are used to obtain simple, accurate formulas for the decision boundaries for sequential probability ratio tests for simple hypotheses and alternatives on the mean θ of the exponential distribution $\theta^{-1}\exp(-u/\theta)$. Examples are provided to indicate the accuracy and the degree of complexity of the results." {From the author's summary. The results of Dvoretzky, Kiefer and Wolfowitz [ibid. **24** (1953), 254–264; MR **14**, 997, 1279] on the Poisson process, which the author mentions as yielding exact formulas for the present problem, have also been used recently to construct tables.} *J. Kiefer.*

Billewicz, W. Z. Matched pairs in sequential trials for significance of a difference between proportions. Biometrics **12** (1956), 283–300.

Wald's method for comparing the means, p and p^*, of two binomial distributions [Sequential analysis, Wiley, New York, 1947, Ch. 6; MR **8**, 593] is modified by the use of paired observations within various strata, the parameters in the ith stratum being p_i and p_i^*. The null hypothesis H_0, that $p_i = p_i^*$, is tested against one of two alternative hypotheses: H_1, that

$$\theta_i = p_i^*(1-p_i)/\{p_i^*(1-p_i)+p_i(1-p_i^*)\} = C > \tfrac{1}{2};$$

and H_1', that $p_i^*/p_i = \lambda > 1$. On H_1 (or H_1'), let θ (or θ') be the proportion of unequal pairs in which a response is obtained in the second series but not in the first. Then, in either situation, pairing within strata, as compared with wholly random pairing, increases θ (θ') and decreases the averages sample numbers in H_0 or H_1 (H_1').

P. Armitage (London).

Žarković, S. S. Einige Bemerkungen über das Problem der relativen Wirksamkeit. Mitteilungsbl. Math. Statist. **8** (1956), 131–140.

The author compares the relative efficiencies of random sampling and stratified sampling (proportional allocation), and observes that if the sample size is determined so that the coefficient of variation of the standard error satisfies a prescribed bound, then in some cases random sampling may be more efficient than stratified sampling. *D. G. Chapman* (Seattle, Wash.).

Bennett, Joseph F. Determination of the number of independent parameters of a score matrix from the examination of rank orders. Psychometrika **21** (1956), 383–393.

Hodnett, G. E. The analysis of a 3×6 experiment arranged in a quasi-Latin square. Biometrics **12** (1956), 245–258.

The analysis of a 3×6 factorial experiment in a 6×6 quasi-Latin square is described and details are given of a general method for computing the standard errors of comparisons between treatment means adjusted for confounding. The application of this method to the computation of standard errors of treatment comparisons in split-plot designs is indicated. *Author's summary.*

Pincus, Howard, J. Some vector and arithmetic operations on two-dimensional orientation variates, with applications to geological data. J. Geol. **64** (1956), 533–557.

See also: Fieller, Lewis and Pearson, p. 422; Geisser, p. 423; Chu, p. 423; Ibers, p. 430; Diederich, p. 439; Chintschin, p. 443; Huber, p. 452; Macey, p. 452; Maximon and Ruina, p. 452.

PHYSICAL APPLICATIONS

Mechanics of Particles and Systems

Viterbo, Francesco. Sul caso più generale di stabilità dei galleggianti (Navi con falle, incagliate, in recupero, ecc.) Ricerca, Napoli **7** (1956), 18–24.

Si semplifica l'espressione del coefficiente di stabilità nelle condizioni più varie applicando un concetto più generale di baricentro e di momento d'inerzia. *Riassunto dell'autore.*

Mangeron, D.; Ciobanu, Gh.; et Braier, Alfred. Sur la distribution des accélérations d'ordre quelconque dans la cinématique des systèmes matériels. Bul. Inst. Politehn. Iaşi (N.S.) **2** (1956), 29–37. (Romanian. Russian and French summaries)

Quelques théorèmes sur l'accélération généralisée dans la cinématique plane dont nous citons: le lieu géometrique des points pour lesquels les accélérations d'ordre n passent par un point donné P est une circonférence de

centre C_n; si P décrit une courbe fermée K, le lieu des centres C_n correspondants à P est une courbe K' semblable à K. Extensions à la cinématique affine et à la cinématique de l'espace. *O. Bottema* (Delft).

Mangeron, D.; Ciobanu, Gh.; et Braier, A. Sur l'extension des formules de Somoff relatives aux accélérations d'ordre supérieur. Rev. Univ. "Al. I. Cuza" Inst. Politehn. Iaşi 2 (1955), 95–100. (Romanian. Russian and French summaries)

Formules recurrentes pour les accélérations d'ordre supérieur dans le mouvement d'un corps rigide. Les accélérations du mouvement composé et leurs projections sur les axes du trièdre de Serret-Frenet. *O. Bottema.*

Mangeron, D.; et Drăgan, Corneliu. Application de la théorie des accélérations réduites d'ordre quelconque à l'étude des mécanismes plans de troisième classe. Bul. Inst. Politehn. Iaşi (N.S.) 2 (1956), 295–304. (Romanian. Russian and French summaries)

Si θ est l'angle de rotation d'une système plane, $A_1=\dot{\theta}^2$, $B_1=\ddot{\theta}$, $A_n=\dot{A}_{n-1}+\dot{\theta}B_{n-1}$, l'accélération réduite est définie a_i/A_i, où a_i est l'accélération d'ordre i. Quelques théorèmes comme le suivant: le lieu géométrique des extrémités des accélérations réduites d'ordre i des points d'une droite d est une droite perpendiculaire à d. *O. Bottema* (Delft).

Manafu, Valeriu. La similitude généralisée dans l'étude des mécanismes. Bul. Inst. Politehn. Iaşi (N.S.) 2 (1956), 305–310. (Romanian. Russian and French summaries)

La représentation par des nombres complexes du champ d'accélérations d'ordre quelconque dans la cinématique plane. Mouvement d'une figure qui reste semblable à elle-même. *O. Bottema* (Delft).

Bondar', N. G. Electric modelling of static stability of systems of rods. Akad. Nauk Ukrain. RSR. Prikl. Meh. 2 (1956), 73–79. (Ukrainian. Russian summary)

★ **Wehrli, Christoph. Kritische Drehzahlen von Wellen mit kurzen Lagern unter konservativer Torsion.** Dissertationsdruckerei Leemann AG, Zürich, 1956. 44 pp.

Diese von der Technischen Hochschule Zürich als Dissertation genehmigte Arbeit behandelt die Berechnung der kritischen Drehzahlen von tordierten Wellen mit kurzen Lagern, die eine Schiefstellung der Biegelinie der Welle erlauben. Wenn durch ein kurzes Lager ein tordierendes Moment eingeleitet wird, soll es als rotierend quasitangential angenommen werden, wie dies der Einleitung durch ein Kreuzgelenk entspricht. Das Moment an der rotierenden Scheibe dagegen wird als semitangential oder ruhend quasitangential angenommen, um die verschiedenen Arten der Einleitung zu erfassen. — Die Analyse der Bewegung des Rotors, der als Schwinger aufgefasst und auf ein raumfestes Koordinantensystem bezogen wird, führt auf eine System von gewöhnlichen linearen Differentialgleichungen mit periodischen Beiwerten. Aus den Eigenschaften der Lösungen des homogenen Systems, die sich an Hand der charakteristischen Gleichung ergeben, wird ein partikuläres Integral des inhomogenen Systems erhalten. Die charakteristischen Exponenten, die für die Stabilität der Lösungen allein massgebend sind, ergeben sich aus dem Nullsetzen einer Determinante mit unendliche vielen Elementen. Eine geeignete Transformation und Entwicklung der Determinante liefert die kritischen Drehzustände für kleine Werte der Torsionsmomente. Ausserdem ermöglicht diese Methode die Berücksichtigung einer Dämpfung die der Geschwindigkeit proportional ist. Mit diesem einfachsten Ansatz zur rechnerischen Verfolgung des Einflusses einer Dämpfung werden die Verhältnisse, wie sie in der Tat vorliegen, nur unvollständig beschrieben.

In der Arbeit werden die Bedingungen der Stabilität für sechs Fälle aufgestellt, und zwar unter Beschränkung auf zylindrische Wellen mit gleichen Biegesteifigkeiten, auf eine einzige rotierende Scheibe sowie auf Belastungen, die im Vergleich zur statischen Knicklast klein sind. In je zwei Fällen wirkt ein semi-tangentiales und ein quasi-tangentiales Moment auf die Scheibe, in zwei weiteren Fällen ist die Scheibe unbelastet. Einer der beiden letzteren Fälle nimmt eine besondere Stellung ein, indem die kritischen Drehzahlen von der gegenseitigen Lage der an den Enden angreifenden, quasitangentialen Momente beeinflusst werden.

R. Gran Olsson (Trondheim).

Vâlcovici, Victor. Une extension des liaisons non holonomes. C. R. Acad. Sci. Paris 243 (1956), 1012–1014.

Une condition $f(q, \dot{q}, t)$ est appelée non-holonome par l'auteur s'il n'est pas possible de la remplacer par une condition équivalente ne contenant pas $\dot{q}$. Il substitue cette définition par une autre qui part d'une condition pour les déplacements infinitésimaux. {Selon l'opinion du ref. cette idée est généralement acceptée dans les cours de mécanique rationelle; voir par exemple Appell, Traité de mécanique rationnelle, 2ième éd., t. I, 1902, pp. 255–256, t. II, 1904, 363–364, Gauthier-Villars, Paris.}

O. Bottema (Delft).

Sideriades, Lefteri. Systèmes non linéaires de deuxième ordre. Applications á l'électronique. C. R. Acad. Sci. Paris 243 (1956), 1850–1852.

Il'inskiĭ, A. Yu. On gliding motions of dynamical systems. Akad. Nauk Ukrain. RSR. Prikl. Meh. 1 (1955), 51–66. (Ukrainian. Russian summary)

A dynamical system described by simultaneous equations (1) $dx_k/dt=X_k$ ($k=1, 2, \cdots, n$), where some of the functions $X_k=X_k(x_1, x_2, \cdots, x_n)$ have discontinuities of the first kind on the hypersurfaces $F_j(x_1, x_2, \cdots, x_n)=0$ ($j=1, 2, \cdots, m$), is discussed as a problem of the trajectory of a point with an initial position $(x_1^0, x_2^0, \cdots, x_n^0)$ at $t=0$. A gliding motion occurs when the point is arrested by the surface of discontinuity and moves with a smaller number of independent coordinates. For study of gliding motions it is suggested by the author to consider equations (1) as a limiting case of another set of equations obtained from (1) by substituting continuous functions X_k^* for X_k, which coincide with X_k everywhere except a narrow region around the discontinuity. An example to this, given by the eqations,

$$dx/d\tau=x+y-m(x);\ dy/d\tau=-x-y,$$

describing a gyroscopic system, is discussed in detail. Here $m(x)=-m$ for $x<0$, and $m(x)=m$ for $x>0$; a continuous function $m^*(x)=-m$ for $x<\varepsilon$; $m^*(x)=m$ for $x>\varepsilon$; and $m^*(x)=mx/\varepsilon$ for $-\varepsilon\leq x\leq\varepsilon$, where ε positive.

S. Kulik (Columbia, S.C.).

Morozova, E. P. Stability of rotation of a solid suspended on a string. Prikl. Mat. Meh. 20 (1956), 621–626. (Russian)

The problem of the title is studied with the aid of a

Lyapunov function, constructed following a method of N. G. Četaev [Prikl. Mat. Meh. **18** (1954), 123–124; MR **15**, 754]. *H. A. Antosiewicz* (Washington, D.C.).

Il'inskiĭ, A. Yu. On the theory of the horizontal gyrocompass. Prikl. Mat. Meh. **20** (1956), 487–499. (Russian)

The essential part of this compass (as in the new Anschütz-Kämpfe gyrocompass) are two gyroscopes with the gyroframe whose centre moves on a immovable sphere surrounding the Earth. The gyroframe represents a mechanical system with four degrees of freedom. Following the elementary gyroscopic theory the relations among the angular momenta, the gyroframe's angular velocity and the moments of the forces are deduced and the relative motion of the frame with respect to the Earth is treated. Using the movable Darboux trihedron, with the origin in the point of suspension of the gyroframe, coupled with the trajectory of this point, the equations of small relative motion of the gyroframe with respect to this trihedron are established. They represent the equations of the forced vibrations of the gyroframe. Assuming small vibrations and supposing that

$$F-mv^2/R\approx F\approx mg,$$

neglecting the friction forces, one obtains a system of four simultaneous linear differential equations with constant coefficients; this is different from the approximative theory of J. W. Geckeler (in 1935, R. Grammel-Der Kreisel, 2. Band). The roofs of the characteristic equation are $\pm i(\nu\pm\omega)$; $\nu=(g/R)^{\frac{1}{2}}$ is the frequency of the M. Schuler period. *D. Rašković* (Belgrade).

Grammel, Richard; und Zieglek, Hans. Der schnelle symmetrische Kardankreisel mit Lagerreibung. Z. Angew. Math. Mech. **36** (1956), 278–279.

Das Kardangehänge sei starr und masselos. Die Bewegungsgleichungen werden für beliebige Reibungsgesetze qualitativ diskutiert. Es zeigt sich z.B. dass die Präzession bei steigendem (sinkendem) Schwerzpunkt rascher (langsamer) ist als beim reibungsfreien Kreisel mit dem gleichen Drehimpuls. Ein ausführlicher Bericht wird angezeigt. *O. Bottema* (Delft).

Magnus, K. Kreiseleigenschaften des umlaufenden Kettenringes. Z. Angew. Math. Mech. **36** (1956), 282–283.

Die Glieder eines Kettenringes sind durch masselose Fäden an eine drehende Scheibe verbunden. Die Schwerkraft wird vernachlässigt, dagegen Widerstandskräfte in Anspruch genommen. Die Bewegungsgleichung eines Kettenelements wird angegeben und diskutiert. Wenn die Kette anfänglich als ebener Kreisring dreht, dessen Ebene einen Winkel mit der Antriebscheibe bildet, so führt sie eine gedämpfte Taumelbewegung aus. Weitere Analogien zwischen dem Verhalten des Ringes und eines Kreisels werden angegeben. *O. Bottema* (Delft).

Preti, Ermenegildo. Fondamenti meccanici del volo gettosostentato. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. **90** (1956), 356–368.

Rudin, Mary Ellen. A separable normal nonparacompact space. Proc. Amer. Math. Soc. **7** (1956), 940–941.

A Hausdorff space having the properties mentioned in the title is constructed by grafting a countable dense set A onto the set B of countable ordinals. The subspace A of X is discrete; the neighborhoods of points of B are complicated, but the closed subspace B reduces to the usual space of countable ordinals. Thus, as the author remarks, X does not have the Lindelöf property. *A. H. Stone* (Manchester).

See also: Reeb, p. 407; Budak, p. 407; Cîmpan, p. 413; Manafu, p. 420; Reed, p. 423; Günther, p. 433; Stanyuković, p. 440; Tesson, p. 442; Gazarhi, p. 447.

Statistical Mechanics

Sack, R. A. A modification of Smoluchowski's diffusion equation. Physica **22** (1956), 917–918.

Rubin, Robert J.; and Shuler, Kurt E. Relaxation of vibrational nonequilibrium distributions. I. Collisional relaxation of a system of harmonic oscillators. J. Chem. Phys. **25** (1956), 59–67.

The problem under consideration is the relaxation of a system of harmonic oscillators which have been excited into a non-equilibrium vibrational state by some external ephemeral interaction. In this paper the means of relaxation are restricted to collisions with other, unexcited particles. The relaxation is described in terms of the collisional probability of Landau and Teller [Phys. Z. Sowjetunion **10** (1936), 34–43]. Thus effects due to unharmonicity and radiative transitions are neglected. The model is explained in terms of its physical counterpart and then the dynamics of the relaxation is derived. The details are given for two initial excitation conditions, namely for a delta function distribution and for a Boltzmann distribution pertaining to a temperature different from that before excitation. The dynamics of the relaxation is quite different for the two distributions. Interestingly the Boltzmann distribution relaxes through a series of quasi-equilibrium Boltzmann distributions characterized by quasi-temperatures intermediate between the initial and final temperatures. Another important result is that the relaxation of the mean energy is the same for the two initial distributions. *M. J. Moravcsik* (Upton, N.Y.).

Rubin, Robert J.; and Shuler, Kurt E. Relaxation of vibrational nonequilibrium distributions. II. The effect of the collisional transition probabilities on the relaxation behavior. J. Chem. Phys. **25** (1956), 68–74.

In this paper which is the continuation of the previous article the effect of the form of the collisional transition probability on the dynamics of the relaxation is explored. The radiative transitions and the effects of unharmonicity are still neglected just as in the first paper. The transitional probability used here is an exponential function of the vibrational quantum number instead of a linear one like in the Landau and Teller form used in the first paper. The behavior with the present transition probability is quite different from the results given in the first paper. Even the relaxation of the mean energy is now dependent on the initial excitation distribution. *M. J. Moravcsik* (Upton, N.Y.).

Jung, H. Vergleich der Berechnungsmethoden für die Druckschwankungen an einer Membran. Hochfrequenztech. Elektroak. **65** (1956), 37–41.

Christov, Chr.; und Nikolov, N. Bemerkung über die Arbeit "Die Auffindung einiger Wahrscheinlichkeiten und Mittelwerte in Bezug auf die Zusammenstösse und die freien Weglängen der Gasmoleküle". Izv. Bŭlgar. Akad. Nauk. Otd. Fiz.-Mat. Tehn. Nauk. Ser. Fiz. **5** (1955), 36a–36b. (Bulgarian. Russian and German summaries)

Christov, Chr.; und Nikolov, N. Die Auffindung einiger Wahrscheinlichkeiten und Mittelwerte in Bezug auf die Zusammenstösse und die freien Weglängen der Gasmoleküle. Izv. Bŭlgar. Akad. Nauk. Otd. Fiz.-Mat. Tehn. Nauk. Ser. Fiz. **5** (1955), 27–36. (Bulgarian. Russian and German summaries)

Cole, G. H. A. On the dynamics of a non-uniform electrically conducting fluid. Nuovo Cimento (10) **4** (1956), 779–785.

In this note the author comments on the construction of a kinetic theory of ionized fluids under non-uniform conditions. *C. H. Papas* (Pasadena, Calif.).

Hooton, D. J. A new treatment of anharmonicity in lattice thermodynamics. I, II. Phil. Mag. (7) **46** (1955), 422–432, 433–442.

These two papers give a modification and extension of the theory of anharmonic vibrations as proposed by M. Born [Festschrift zur Feier des zweihundertjährigen Bestehens der Akad. Wiss. Göttingen, I, Math.-Phys. Kl., Springer, Berlin, 1951, pp. 1–16; Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. IIa. **1951**, no. 6; Changements de phases, Soc. Chim. Phys., Paris, 1952, pp. 334–337]. Large anharmonic effects arise for certain materials, particularly solid helium, if the quantum mechanical "zero-point energy" already gives such large amplitudes of oscillation that the usual harmonic approximation to the potential energy is inaccurate. The present treatment assumes an expansion of the potential energy to include third and fourth degree terms in the displacements. Instead of using a perturbation method, the author chooses a variable set of frequencies and harmonic oscillator coordinates measured from the mean particle positions rather than from the potential minimum. The free energy of the system is evaluated approximately as a function of these coordinates and the coordinates and frequencies are finally chosen so as to minimize the free energy. In effect then, one obtains for any conditions of temperature volume, etc. a "best" harmonic oscillator approximation.

The new method gives results consistent with the perturbation treatment of small anharmonic effects; it is also applicable for large effects although the accuracy of certain approximations may not be apparent.

The first paper describes the general formalism and the second contains a specialization in which a Debye continuum approximation is made along with special assumptions regarding the form of the anharmonic terms. *G. Newell* (Providence, R.I.).

Ibers, James A. Estimates of the standard deviations of the observed structure factors and of the electron density from intensity data. Acta Cryst. **9** (1956), 652–654.

In typical crystallographic analyses each reflexion is in general measured twice, in the course of rotations about two crystallographic axes, say a and b. The mean value and the standard deviation of the corresponding structure amplitude may then be estimated as

$$F_0=\tfrac{1}{2}(F_a+F_b),$$
$$\sigma(F_0)=0.89|F_a-F_b|.$$

Certain reflexions may be measured only once; the standard deviation must then be assumed equal to that of reflexions of the same magnitude measured twice. F's too small to be observed have mean values and standard deviations calculable from the appropriate statistical distribution; F's identically zero for space-group reasons have standard deviations identically zero. With these estimates of standard deviations, the standard deviation of the electron density can be estimated from Cruickshank's [Acta Cryst. **2** (1949), 65–82; **3** (1950), 72–73] expression

$$\sigma(\rho_0)=mV^{-1}[\sum\{\sigma(F_0)\}^2]^{1/2},$$

where the sum is over all reflexions, V is the volume of the unit cell of the crystal, and m is 1 for centrosymmetric space groups and 2 for non-centrosymmetric ones. This estimate of $\sigma(\rho)$ can be evaluated whether or not the signs (phases) of the F's are known, and hence $\sigma(\rho)$ can be estimated before attempting a calculation of ρ.

Applications and limitations are discussed.

A. J. C. Wilson (Cardiff).

Raman, C. V. The specific heats of crystals. I. General theory. Proc. Indian Acad. Sci. Sect. A. **44** (1956), 153–159.

Raman, C. V. The specific heats of crystals. II. The case of diamond. Proc. Indian Acad. Sci. Sect. A. **44** (1956), 160–164.

See also: Bilby and Smith, p. 430; Pai, p. 437; Laitone, p. 440; Keller, p. 442.

Elasticity, Visco-elasticity, Plasticity

Vening Meinesz, F. A. Elasticity and plasticity. Appl. Sci. Res. A. **6** (1956), 205–225.

Bilby, B. A.; and Smith, E. Continuous distributions of dislocations. III. Proc. Roy. Soc. London. Ser. A. **236** (1956), 481–505.

Als Fortsetzung einer früheren Arbeit der Verfasser [Bilby, Bullough und Smith, derselben Proc. **231** (1955), 263–273; MR **17**, 687] wird die Theorie des kontinuirlich versetzten Kristalles erweitert. Es wird gezeigt, dass die Gleichungen von Nye — von dem der Gedanke herrührt, dass man die Versetzungen des selbstverständlich atomistisch aufgebauten Kristalles verschmiert, um auf diese Weise die mathematische Behandlung zu vereinfachen — nur dann anwendbar sind, wenn die Krümmungen des Kristalls (also die Dichten der Versetzungen) klein sind. Die strengen Gleichungen für starke Krümmungen werden hergeleitet, wobei eine nichtriemannsche Geometrie, ebenso wie in der zitierten Arbeit, benutzt wird. Zwischen den Rotationskoeffizienten von Ricci und dem hier eingeführten Torsionstensor — welcher die lokale Verteilung der Versetzungen beschreibt — wird ein Zusammenhang hergeleitet.

In der vorliegenden Arbeit werden eigentlich nur die Zusammenhänge zwischen den inneren Spannungen und der Geometrie des Gitters, welche die Verteilung der Ver-

setzungen beschreibt, besprochen. Ein wirklicher Kristall wird jedoch auch gleiten, zu fliessen beginnen und sich dabei verfestigen. Diese Fragen sollen in einer künftigen Arbeit der Verfasser besprochen werden.

T. Neugebauer (Budapest).

Hill, R. New horizons in the mechanics of solids. J. Mech. Phys. Solids **5** (1956), 66–74.

This is the text of a General Lecture delivered at the 9th International Congress of Applied Mechanics (Brussels, September, 1956). The main part of the paper contains a systematic presentation of uniqueness theorems and extremum principles for a large variety of solids, the unifying concept being that of a convex function ("work function" or "complementary work function"). The less systematic remainder of the paper contains stimulating examples of exact and partial correspondences between various types of solids, and some discussion of the possibility of removing indeterminacy by treating the solid in question as a special case of a solid of a more general type.

W. Prager (Providence, R.I.).

Il'yušin, A. A. On a connection between stresses and small strains in the mechanics of continuous media. Prikl. Mat. Meh. **18** (1954), 641–666. (Russian)

An extremely detailed mathematical treatment is given of stress-strain relationships in a total strain theory of plasticity. This treatment is directed towards the establishment of results of a geometrical nature, and use is made of the spinor calculus. The second half of the paper is devoted to the specialization of the general analysis to the particular case of isotropic material and then to further specialization to conditions of plane stress.

H. G. Hopkins (Sevenoaks).

Koppe, Eberhard. Die Ableitung der Minimalprinzipien der nichtlinearen Elastizitätstheorie mittels kanonischer Transformation. Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. IIa. **1956**, 259–266.

This is a neat, but poorly documented, exposition of variational principles applicable to static problems in nonlinear elasticity.

J. L. Ericksen.

Le Boiteux, Henri; et Pauthier, Suzanne. Comportement général des matériaux visqueux dans les domaines élastique et plastique. Rech. Aéro. no. **54** (1956), 23–30.

Colonnetti, Gustavo. L'équilibre des voiles minces hyperstatiques. C. R. Acad. Sci. Paris **243** (1956), 761–764.

The author applies his variational postulate for elastic-plastic bodies [J. Math. Pures Appl. (9) **33** (1954), 187–199; MR **16**, 308] to thin sheets.

C. A. Truesdell.

Vlasov, V. Z. Theory of pre-stressed thin-walled rods, plates and shells. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1956**, no. 5, 70–84. (Russian)

Levin, M. L.; and Rytov, S. M. On the transition to the geometrical approximation in the theory of elasticity. Akust. Ž. **2** (1956), 173–176. (Russian)

★**Sekiya, Tsuyoshi. An approximate solution in the problems of elastic plates with an arbitrary external form and a circular hole.** Proceedings of the Fifth Japan National Congress for Applied Mechanics, 1955, pp. 95–98. Science Council of Japan, Tokyo, 1956.

In this paper a method is described for the approximate solution of problems of plane stress or plane strain and of transverse bending of thin isotropic plates in domains which have an arbitrary external boundary and a circular hole. For the plane stress and plane strain problems the author quotes results of a previous paper [Negoro and Sekiya, Proc. 1st Jap. Nat. Congress Appl. Mech., 1951, p. 179–184] and uses them for the particular case of a square plate with a circular hole. The bending of a uniformly loaded square plate with a circular hole is also treated. In both cases the approximate nature of the solutions depends on satisfying the exact boundary conditions only at a finite number of points on the outer boundary.

R. M. Morris (Cardiff).

Chatterjee, B. B. Stresses in a rotating blade bounded by two equal confocal parabolas. Indian J. Theoret. Phys. **3** (1955), 107–110.

In this paper a successful attempt has been made to find stresses in a rotating blade bounded by two equal confocal parabolas when the blade rotates steadily in its own plane about a normal axis. The problem has been dealt with as one of generalised plane stress in which body forces are present. The reversed mass-accelerations in steady rotation have been treated as body forces.

The method of solution given by A. C. Stevenson [Philos. Mag. (7) **34** (1943), 766–793; MR **8**, 116] has been employed. Two cases are considered: The blade rotates 1) about an axis through the origin, and 2) about an axis at a distance equal to the radius of the blade from the origin. Since in the latter case the blade is not symmetrical about the axis of rotation, a centrifugal force is developed at the point in the blade through which the axis of rotation passes. This force is, however, counteracted if a similar blade is taken at the other side of the axis of rotation.

R. Gran Olsson (Trondheim).

Teodorescu, Petre P. Une méthode pour résoudre le problème plan de la théorie de l'élasticité dans le cas de forces massiques quelconques. Com. Acad. R. P. Romîne **6** (1956), 285–290. (Romanian. Russian and French summaries)

The author obtains formal solutions to two-dimensional problems of elasticity theory for arbitrary action of mass-forces. The stress-strain components are expressed in terms of biharmonic functions (also non-homogeneous). {These expressions are, however, of mere theoretical interest. No specific solution is given for any boundary-value problem. The author does not seem to have heard of Mushelišvili's works [cf. Singular integral equations, Gostehizdat, Moscow-Leningrad, 1946; MR **8**, 586; **15**, 434].}

K. Bhagwandin (Oslo).

Griffith, J. E.; and Marin, Joseph. Creep relaxation for combined stresses. J. Mech. Phys. Solids **4** (1956), 283–293.

The authors first extend the usual assumption of simple tension creep, viz. that the deformation is made up of an elastic, a plastic, a transiant and a steady state (constant creep rate) component to determine the constant stress strain-stress-time relations for combined stresses. This relation is used in developing a theory for predicting stress relaxation-time relations for creep under combined stresses — according to the authors, the first such attempt. The accuracy of the theoretical predictions is checked experimentally by creep relaxation tests on a thin-walled circular aluminium tube under axial tension and torsion; these indicate approximate agreement with the theory.

W. Freiberger (Toronto, Ont.).

Kroupa, František. The mixed boundary value problem of the plane theory of elasticity for an annular region. Czechoslovak J. Phys. 6 (1956), 124–140. (Russian. English summary)

Kostandyan, B. A. On torsion of a hollow step shaped beam. Akad. Nauk Armyan. SSR. Izv. Fiz.-Mat. Estest. Tehn. Nauki 9 (1956), no. 3, 17–32. (Russian. Armenian summary)

The stress function for torsion is determined for the solid of revolution

$$s<r<d \ (0<z<a),$$
$$s<r<R \ (a<z<b),$$

where $d>R$. The stresses $\tau_{\xi\theta}$ are given by Fourier-Dini series with known coefficients on the two ends $z=0$, $z=b$ and the remaining portions of the boundary are free. The stress function Φ is assumed to be given by different expressions Φ_1, Φ_2, Φ_3 in the three regions

I. $s<r<R,\ a<z<b$;
II. $s<r<R,\ 0<z<a$;
III. $R<r<d,\ 0<z<a$.

The stresses across the boundaries between I, II and between II, III are given by series with unknown coefficients

$$\tau_{r\theta}=\tfrac{1}{2}\eta_0+\sum_1^\infty \eta_k \cos\frac{k\pi z}{a} \ (r=R,\ 0<z<a),$$

$$\tau_{z\theta}=\xi_0 r+\sum_1^\infty \xi_k\omega_1(\lambda_k r) \ (s<r<R,\ z=a),$$

the functions $\omega_1(\lambda_k r)$ being certain combinations of Bessel functions. Making the functions Φ_1, Φ_2 agree on their common boundary (also Φ_2, Φ_3) leads to infinite sets of equations for the ξ_k, η_k. These systems of equation are shown to be completely regular.

Some numerical results are given. *R. C. T. Smith.*

Lehnickiĭ, S. G. Torsion of a many-layer rod of rectangular cross-section. Inžen. Sb. 23 (1956), 63–76. (Russian)

A composite rectangular beam with n layers of orthotropic material, the kth of thickness h_k-h_{k-1}, is considered. The boundary conditions for each layer are taken in the form

$$\tau_{xz}=0 \ (x=0,\ x=a),$$

$$\tau_{yz}=\tau_{k-1}=\sum_{1,3,5,\cdots}\tau_{k-1,m}\cos\frac{m\pi x}{a} \ (y=h_{k-1}),$$

$$\tau_{yz}=\tau_k=\sum_{1,3,5,\cdots}\tau_{k,m}\cos\frac{m\pi x}{a} \ (y=h_k),$$

where $\tau_{0,m}=\tau_{n,m}=0$ $(m=1, 3, 5, \cdots)$. The stress and displacement functions are determined by a routine calculation in terms of the $\tau_{k,m}$. Finally making the displacements on each side of the surface $y=h_{k-1}$ agree leads to recurrence relations in $\tau_{k,m}$, $\tau_{k-1,m}$, $\tau_{k-2,m}$.

The case of a symmetrical beam with three layers of equal thickness is treated in detail. *R. C. T. Smith.*

Budiansky, Bernard; and Mayers, J. Influence of aerodynamic heating on the effective torsional stiffness of thin wings. J. Aero. Sci. 23 (1956), 1081–1093, 1108.

Becker, Herbert; and Gerard, George. Torsional buckling of moderate-length cylinders. J. Appl. Mech. 23 (1956), 647–648.

Satō, Sennosuke. Maximum torsional stress in splined and serrated shafts. J. Appl. Mech. 23 (1956), 648–649.

Ščetinin, N. N. Pure bending of rods in the case of creeping of the material. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk 1956, no. 8, 37–41. (Russian)

Pure bending with creep of a rectangular bar, sides $2b$, $2h$, under a constant moment M is considered. The strain-time law is

$$p^{\nu-1}\frac{\partial p}{\partial t}=n\left(\exp\frac{\sigma}{A_0}-\right)1,$$

where $p=\varepsilon-\sigma/E$, ν, n, A_0, E being constants for the material at a given temperature. The other assumptions are $p=0$ for $t=0$, $\varepsilon=\chi(t)y$, $4b\int_0^h \sigma y dy=M$. Convergent series for p and σ as functions of t are obtained.

R. C. T. Smith (Armidale).

Nudel'man, Ya. L.; and Ovčinnikov, P. F. Bending of rods of variable cross-section taking into account the shear deformation. Akad. Nauk Ukrain. RSR. Prikl. Meh. 2 (1956), 40–50. (Ukrainian. Russian summary)

This is a non-rigorous engineering treatment of the problem. A term depending on the variation of the cross-section is included in the expression for the average shear stress [see B. L. Nikolai, Vestnik Inž. Meh. 1939 no. 7, 304–307]. The strain energy due to the shear and bending stresses is calculated and differential equations set up by a variational method. Some examples are treated which show that the additional term may be of some importance.

R. C. T. Smith (Armidale).

Abbassi, M. M. The mathematical analysis of bow girders of any shape. J. Appl. Mech. 23 (1956), 522–526.

By using parametric equations in which the parameter is the angle included between the tangent at any point on the bow girder and the tangent at the middle point, the analysis of bow girders of shapes other than the circular arc can be treated mathematically. Exact and approximate formulas are given for symmetrical bow girders of any shape carrying a distributed load or two equal concentrated loads placed symmetrically with respect to the middle point of the girder. *Author's summary.*

Schwieger, H.; und Haberland, G. Spannungsoptische Untersuchungen von Platten mit veränderlicher Dicke. Z. Angew. Math. Mech. 36 (1956), 287–288.

Mit Hilfe spannungsoptischer Methoden wird der Spannungszustand gebogener Platten mit veränderlicher Dicke ermittelt. Als Versuchsobjekt wurde eine quadratische Platte mit einem eingespannten Rand und drei freien Rändern sowie linear veränderlicher Dicke gewählt, da bei diesen Randbedingungen keine exakte Lösung bekannt war. Aus den zu beobachtenden Isoklinen und Isokromaten liessen sich die Linien der Hauptbiegemomente und die Linien gleicher Hauptdrillungsmomente konstruieren. Um die einzelnen Hauptträgemomente zu ermitteln, wurde eine besondere Methode der Integration abgeleitet, die aus den spannungsoptischen Spezifikationen die Ersatzmomente sowie die Hauptbiegemomente zu berechnen gestattete.

R. Gran Olsson (Trondheim).

Kączkowski, Zbigniew. Orthotropic rectangular plates with arbitrary boundary conditions. Arch. Mech. Stos. 8 (1956), 179–196.

Fung, Y. C., and Wittrick, W. H. A boundary layer phenomenon in the large deflexion of thin plates. Quart. J. Mech. Appl. Math. 8 (1955), 191–210.

The authors begin with a discussion of the problem of pure bending of a this rectangular plate. They show that in the large deflection range the solution is composed of two parts. One part applies except in a narrow edge zone and is a state of cylindrical bending without extension of the middle surface. The other part is effectively limited to just the narrow edge zone and has certain important qualitative features. In generalisation of this the authors consider other problems which have similar properties and for these they simplify the problem of solving the equations of large deflection theory as follows. Given the system

$$D\nabla^4 w = p(x) + N_x w_{,xx} + N_y w_{,yy} + 2N_{xy} w_{,xy},$$
$$\nabla^2(N_x + N_y) = C[w_{,xy}{}^2 - w_{,xx} w_{,yy}],$$

it is shown that the assumption $w = w^*(x) + \zeta(x, y)$ allows an approximate reduction to the simplified form

$$Dd^4w^*/dx^4 = p,\ D\partial^4\zeta/\partial y^4 = N_x \partial^2 w^*/dx^2,$$
$$\partial^2 N_x/\partial x^2 = -C(d^2w^*/dx^2)\partial^2\zeta/\partial y^2,$$

where w^* corresponds to the interior state and ζ, which is effectively determined from an equation of the form $\partial^4\zeta/\partial y^4 + 4\alpha^4\zeta = 0$ corresponds to the edge zone state.

As an application of their theory the authors consider a rhombic plate loaded by two pairs of equal and opposite corner forces. Knowing the separation into edge zone state and interior state the authors determine an effective load deflection ratio corresponding to the interior state, as a function of rhombus angle. The relation between load and deflection which they obtain is linear but applies only in the large-deflection range of the theory. A comparison is made with a corresponding result by the reviewer, which is valid in the small deflection range [Quart. Appl. Math. 10 (1953), 395–397; MR 14, 429]. (It remains to obtain a solution for the problem in the intermediate range in which the transition from one of the linear relations to the other takes place.) The paper concludes with some observations on the role of inextensional (or nearly inextensional) bending in the buckling of cylindrical shells and on inextensional plate bending which is not cylindrical.

E. Reissner (Cambridge, Mass.).

Mansfield, E. H. The inextensional theory for thin flat plates. Quart. J. Mech. Appl. Math. 8 (1955), 338–352.

It is observed that the bending of cantilever plates, in the large-deflection range, is essentially without middle surface extension. This means that the transverse deflection w satisfies the differential equation $w_{,xy}{}^2 - w_{,xx}w_{,yy} = 0$ or in other words that the deflection surface is a developable surface. The author's procedure is to start with a parametric equation for developable surfaces. In view of the fact that there is just one surface parameter and not two as for more general classes of surfaces it turns out that the determination of the solution surface becomes an ordinary-differential-equation problem. The actual form of the appropriate differential equation is obtained by means of strain energy considerations. Applications of the method are carried out for "swept" plates acted upon by end moments and triangular plates acted upon by a transverse force at apex.

E. Reissner.

Nazarov, A. A. On large bendings and stability of a slanting shell of double curvature whose edges are rigidly fixed. Dopovidi Akad. Nauk Ukrain. RSR **1956**, 231–234. (Ukrainian. Russian summary)

The following generalization of the von Karman large deflection equations to shells with principal directions of curvature parallel to the x, y axes is considered.

$$(*)\quad \nabla^2\nabla^2 F = E\left\{\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - r_0\frac{\partial^2 w}{\partial y^2} - t_0\frac{\partial^2 w}{\partial x^2}\right\};$$

$$(**)\quad \nabla^2\nabla^2 w =$$
$$\frac{1}{D}\left\{P + h\left[\left(r_0 + \frac{\partial^2 w}{\partial x^2}\right)\frac{\partial^2 F}{\partial y^2} + \left(t_0 + \frac{\partial^2 w}{\partial y^2}\right)\frac{\partial^2 F}{\partial x^2} - 2\frac{\partial^2 F}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial y}\right]\right\}.$$

For clamped edge conditions w may be expressed in the form

$$w = \sum_1^\infty \sum_1^\infty C_{mn}\left(1 + \cos\frac{2\pi(2m-1)}{a}x\right)\left(1 + \cos\frac{2\pi(2n-1)}{b}y\right)$$

and (*) solved for F. If the expressions for F and w are substituted in (**) it is possible to solve for the C_{mn} approximately by Gal'erkin's method. A first approximation for the case of buckling under normal pressure is obtained from a single term in the series for w.

R. C. T. Smith (Armidale).

Nazarov, A. A. On large deflections and stability of a sloping shell of double curvature supported on all edges. Dopovidi Akad. Nauk Ukrain. RSR **1956**, 349–352. (Ukrainian. Russian summary)

Problem considered is that of a doubly curved thin shell with rectangular plan form, under the action of normal pressure and membrane forces. The edges of the shell are simply-supported and are not free to move. The basic differential equations, in terms of the normal deflection and a stress function for the membrane forces are solved in terms of double Fourier series. By considering the first few terms of these series, an expression is obtained for the normal pressure in terms of the deflection of the center of the shell and the ratio δ/h, where δ is the maximum height of the shell above its plan form, and h is the thickness of the shell. Based on this last equation, a table is given for the values of δ/h for which the equilibrium equation breaks down, and for which instability can occur. Results are compared to similar ones for a cylindrical shell.

H. P. Thielman and H. J. Weiss.

Günther, W. Gleichgewicht und Verträglichkeit in der Schalenbiegetheorie. Z. Angew. Math. Mech. 36 (1956), 279–280.

In order to determine the twelve force and moment components acting on the edges of a shell element, six equilibrium conditions are available. The equilibrium problem of the shell is therefore six-fold functionally redundant. It is shown that the six arbitrary stress functions that appear in the solution, can be assembled in two vector fields, the "vectorial force potential" and the "vectorial moment potential".

After introducing a distortion vector and a curvature vector, the principle of virtual forces is used to derive the displacement-deformation equations and the compatibility equations for the deformations. There exists a formal similarity between the representation of the force and moment components by the stress functions and the representation of the deformations by the displacements,

and also between the equilibrium conditions for the force and moment components and the equations of compatibility for the deformations.

The results are discussed in connection with some special cases. *H. D. Conway* (Ithaca, N.Y.).

Vorovič, I. I. On certain direct methods in the non-linear theory of sloping shells. Prikl. Mat. Meh. **20** (1956), 449–474. (Russian)

A very condensed version of this article appeared earlier in [Dokl. Akad. Nauk SSSR (N.S.) **105** (1955), 42–45; MR **17**, 685]. *R. C. T. Smith* (Armidale).

McComb, Harvey G., Jr. Torsional stiffness of thin-walled shells having reinforcing cores and rectangular, triangular, or diamond cross section. NACA Tech. Note no. **3749** (1956), 35 pp.

Problem concerns a hollow thin-walled shell filled with a material which is bonded to the shell. Exact solution is given for a rectangular shell and this gives St. Venant's solution for a rectangular bar as a special case. The two other problem are solved by Rayleigh-Ritz and variational methods, numerical results being presented graphically. *H. D. Conway* (Ithaca, N.Y.).

Sigua, F. D. On a boundary problem for a thin spherical shell. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze **22** (1956), 265–275. (Russian)

Žgenti, V. S. On the properties of the solution of an elastic thin shell which has the shape of an elliptic paraboloid. Soobšč. Akad. Nauk Gruzin. SSR **17** (1956), 193–200. (Russian)

Frederick, Daniel. Physical interpretation of physical components of stress and strain. Quart. Appl. Math. **14** (1956), 323–327.

For a vector, the author regards as physical components any one of the four sets of projections mentioned by Ricci and Levi-Cività. To get physical components for the stress tensor, the author considers the stress vectors associated with the covariant or the contravariant base triad and resolve each as above; this gives eight sets of physical components, some of which, as the author remarks, have been introduced by previous writers. An entirely different procedure is used for the strain. *C. A. Truesdell* (Bologna).

Arakelyan, T. T. Bending of an infinite beam based on a continuous earth foundation. Akad. Nauk Armyan. SSR. Izv. Fiz.-Mat. Estest. Tehn. Nauki **9** (1956), no. 3, 45–61. (Russian. Armenian summary)

An infinite elastic beam with a concentrated vertical load at the origin acting from time $t=0$ onwards and supported by a layer of yielding water-saturated earth is considered. Using the expression relating the deflection of the surface of such a layer to the normal pressure [Arakelyan, same Izv. **4** (1953), no. 2, 55–71] an integro-differential equation for the reaction between beam and foundation is set up. This equation is solved by separation of variables. *R. C. T. Smith* (Armidale).

Zanaboni, Osvaldo. Sulla deformazione indotta dallo sforzo tagliante. Atti Accad. Sci. Ist. Bologna. Cl. Sci. Fis. Rend. (11) **1** (1953–54), no. 1, 13–22.

This is an expository paper dealing with elastic deformations induced by shear forces, within the framework of the elementary theory of structures and from a kinematic point of view. *F. B. Hildebrand.*

Cox, Hugh L.; and Klein, Bertram. Vibration of isosceles triangular plates having the base clamped and other edges simply-supported. Aero. Quart. **7** (1956), 221–224.

In the present paper approximate solutions are given for the lowest natural frequency of flexural vibration of isosceles triangular plates which have the equal edges simply supported and the base clamped. The solutions are obtained by using the method of collocation [see E. G. Keller, Mathematics of modern engineering, v. 2, Wiley, New York, 1942, p. 285; MR **4**, 150; B. Klein and H. L. Cox, J. Aero. Sci. **21** (1954), 719]. The numerical results are presented graphically to allow rapid determination of a desired fundamental frequency. *R. Gran Olsson.*

Choudhury, Pritindu. Two-dimensional problems of stress distribution due to certain loads on the upper surface of an elastic layer of non-isotropic material with rigid base. Indian J. Theoret. Phys. **3** (1955), 111–118.

This paper is mainly concerned with the two-dimensional problems of an elastic layer resting on a rigid base, the material of the layer having a simple form of aeolotropy. Three types of normal loads on the upper surface, namely loads having distribution of Gaussian, parabolic and rectangular types over a finite portion of the surface, have been considered. The layer having the aeolotropy of the type discussed in this paper has not been considered before. The problems considered are solved by using Fourier transforms.

Problems of an elastic layer of isotropic material resting of an elastic foundation have been discussed previously by K. Marguerre [Ing.-Arch. **2** (1931), 108–117] and M. A. Biot [Physics **6** (1935), 367–375]. (There are some minor misprints, e.g. on p. 117.) *R. Gran Olsson.*

Savin, G. N.; Ševelo, V. N.; and Kužil, A. I. On longitudinal vibrations of a thread of variable length taking into account internal friction of hysteresis type. Akad. Nauk Ukrain. RSR. Prikl. Meh. **2** (1956), 133–146. (Ukrainian. Russian summary)

Chakravorty, J. G. Vibrations of spherically aeolotropic shell. Bull. Calcutta Math. Soc. **47** (1955), 235–238.

Very few problems of spherical aeolotropic elastic material have been considered so far because of the inherent difficulty in solving complicated simultaneous partial differential equations. B. de Saint-Venant has considered the equilibrium of a spherically aeolotropic elastic shell under internal and external pressure [J. Math. Pures Appl. (2) **10** (1865), 297–349] and Seth has extended the solution to the case of finite strain [Bull. Calcutta Math. Soc. **37** (1945), 62–68; **38** (1946), 39–44; MR **7**, 143, 501].

In the present paper two problems of vibration of spherical shell of spherically aeolotropic material have been considered, viz. 1) radial vibration and 2) rotary vibration of the shell. In the solution of both these problems as well as those of Saint-Venant and B. R. Seth the success is due to the fact that only one of the components of displacement is non-zero, so that two of the equations of motion or equilibrium are identically satisfied whereas the third reduces to an ordinary differential equation of the second order, the solution of which is given by Bessel functions. *R. Gran Olsson* (Trondheim).

Lebedev, N. F. The repeated elastic-plastic wave. Prikl. Mat. Meh. **18** (1954), 167–180. (Russian)

Theoretical analysis is given of a problem of longitudinal, plastic wave propagation in a bar exhibiting strain-hardening mechanical behaviour. The primary plastic wave is caused by sudden application of stress at one end of the bar. The author's main concern is with the study of the secondary plastic wave caused by reflection of the primary plastic wave at the free end of a finite bar. The usual method based upon characteristics theory is employed. *H. G. Hopkins* (Sevenoaks).

Lebedev, N. F. Propagation of an impact wave in a semi-infinite uniform bar. Inžen. Sb. **11** (1952), 103–122. (Russian)

Analysis, based upon characteristics theory, is given of the propagation of loading and unloading longitudinal plastic waves in a semi-infinite bar exhibiting work-hardening mechanical behaviour. *H. G. Hopkins.*

Rahmatulin, H. A., and Šapiro, G. S. Propagation of disturbances in a nonlinearly elastic and a nonelastic medium. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1955**, no. 2, 68–89 (1 plate). (Russian)

This paper is mainly an account of the literature published on the subject of non-linear wave propagation in solids. Thus, attention is given to the following types of mechanical behaviour: non-linear elastic, elastic-plastic, visco-elastic, visco-plastic and visco-elastic-plastic. A very full bibliography with 77 references is appended to the paper. *H. G. Hopkins* (Sevenoaks).

Philipson, L. L. On the role of extension in the flexural vibrations of rings. J. Appl. Mech. **23** (1956), 364–366.

Author investigates validity of nonextension assumption in flexural vibrations of a circular ring segment by generalizing Love's classical analysis. He shows that the form of Love's equilibrium equations is still correct if extension is permitted but that the stress components and couples in the equations have different expressions in terms of the displacements. The forced vibration problem is also treated. *H. D. Conway* (Ithaca, N.Y.).

Jones, R. P. N. The reflection of transverse waves in beams. Quart. J. Mech. Appl. Math. **9** (1956), 499–507.

Shibaoka, Yoshio. On the transverse vibration of an elliptic plate with clamped edge. J. Phys. Soc. Japan **11** (1956), 797–803.

The author states, that N. W. MacLachlan's previous solution of this problem [Quart. Appl. Math. **5** (1947), 289–297; Theory and application of Mathieu functions, Oxford, 1947; MR **9**, 316, 31] is in error, as he assumed that one normal mode of vibration may be expressed by one particular solution of the differential equation, whereas it may only be expressed by an infinite sum of such solutions. The author derives the suitable Mathieu differential equations of the problem, which are solved by Mathieu functions of the first and the second kind. By symmetry, only symmetric functions may be used in the composition of a solution, restriction being made to those normal modes of vibration, which are of symmetric displacement about both axes. The appropriate constant multipliers of the solutions are found by series expansion of Mathieu functions. Finally, an infinite determinantal equation is found for the eigenvalues, in which the determinant may be expanded in a series of finite ones. Numerical values are given for the normal vibration frequencies in several cases. In the case, that the elliptic plate differs only slightly from a circular one, the fundamental normal frequency is expanded in a power series of the excentricity. *M. J. O. Strutt* (Zurich).

Lücke, Kurt. Ultrasonic attenuation caused by thermoelastic heat flow. J. Appl. Phys. **27** (1956), 1433–1438.

Chilver, A. H. Corrected discontinuities in structural stability problems. J. Mech. Phys. Solids **5** (1956), 9–17.

The basic differential equations of a structural stability problem are derived usually by one of two methods. The first method, described by the author as the "conventional" approach, consists in considering the equilibrium and strain-compatibility of an isolated element of the structure. The second method, called the "energy" approach, consists in establishing the equation of total potential energy of the system and minimizing this expression. The conventional method becomes involved in dealing with complex problems, e.g., torsional-flexural buckling of a compressed slender column. The energy method offers a general approach to the solution of all elastic stability problems.

The approach discussed in this paper makes use of the concept of corrected discontinuities. This involves an analysis similar to the conventional method, but it has the great advantage that the basic differential equations for certain problems may be derived more easily. In this modified method the structure is reduced initially to a system of disconnected elements. For any buckled shape the external forces required for equilibrium of the elements are evaluated. A second system of external forces is then found to give a continuous structure with the same displaced shape. If the two systems of external forces are in equilibrium the buckled form can be maintained without external forces, and this corresponds to the required condition of neutral equilibrium. The method consists essentially in correcting an arbitrary system of discontinuities, and is discussed in detail on the problem of the torsional-flexural stability of a thin-walled open-section axially loaded strut. *R. Gran Olsson.*

Koiter, W. T. On the flexural rigidity of a beam, weakened by transverse saw cuts. I, II. Nederl. Akad. Wetensch. Proc. Ser. B. **59** (1956), 354–364, 365–374.

Der Rotor zweipoliger Turbogeneratoren hat eine reduzierte Biegesteifigkeit in Bezug auf eine der Hauptachsen wegen der längsgehenden Schlitze, die für die elektrischen Windungen vorgesehen sind. Dies mag eine sekundäre kritisiche Geschwindigkeit etwa gleich der Hälfte der normalen kritischen Geschwindigkeit herbeiführen. J. Dispaux hat vorgeschlagen [Electrotechnik **33** (1955), 291–300], diese Schwierigkeit durch Herstellung gleicher Biegesteifigkeit um beide Hauptachsen mit Hilfe quergehender Schnitte an den Pol-Zylinderflächen zu überwinden. Diese Lösung hat keinen Einfluss auf den magnetischen Strom, und die Spannungskonzentration an der Wurzel der Schnitte bildet kein Problem, weil die Biegespannungen klein sind.

Ein theoretischer Zugang zum Problem kann durch Untersuchung des entsprechenden ebenen Biegeproblems eines unendlichen Streifens, der durch eine unendliche Anzahl von gleichen und äquidistanten Schnitten senkrecht zum Rand geschwächt ist. Die Spannungsverteilung

und Steifigkeit des geschwächten Streifens hängen von zwei Parametern b/H und b/h ab, wo b die Teilung der Schnitte, H die wirksame Höhe und h die Höhe der Schnitte bedeuten (es ist also $h=D-2h$, wo D=Durchmesser). Es wird eine der Kerbtiefe h „äquivalente Höhe" h_e eingeführt, definiert durch die Höhe $(H+2h_e)$ eines ungekerbten Streifens mit der gleichen Biegesteifigkeit wie der gekerbte Streifen von der Höhe $(H+2h)$. Das ebene Spannungsproblem ist gelöst, wenn das Verhältnis h_e/h als Funktion der Parameter b/H und b/h ermittelt ist. Das Hauptergebnis der Analyse ist durch die Formel gegeben:

$$h_e=\tfrac{1}{2}H\left[\left(1+6\alpha_{11}\frac{h}{H}+12\alpha_{12}\frac{h^2}{H^2}+8\alpha_{22}\frac{h^3}{H^3}\right)^{1/3}-1,\right]$$

wo die dimensionslosen Koeffizienten α_{11}, α_{12}, α_{22} nur vom Parameter $\beta=b/h$ abhängen, falls der Parameter b/H hinreichend klein angenommen wird $(b/H\approx 1)$. Die Koeffizienten α sind graphisch als Funktionen von β dargestellt. *R. Gran Olsson* (Trondheim).

Mise, K.; and Kunii, S. A theory for the forced vibrations of a railway bridge under the action of moving loads. Quart. J. Mech. Appl. Math. 9 (1956), 195–206.

Chilver, A. H. A note on the Mise-Kunii theory of bridge vibrations. Quart. J. Mech. Appl. Math. 9 (1956), 207–211.

Chilver, A. H. Buckling of a simple portal frame. J. Mech. Phys. Solids 5 (1956), 18–25.

E. Chwalla has shown [Bauingenieur 19 (1938), 69–75] that simple portal frames may have buckling characteristics somewhat different from those of a single strut. Chwalla found that when vertical loads are applied to the cross beam of a portal frame the critical value of these loads for lateral instability of the whole frame may be slightly smaller than the critical value of loads applied to the tops of stanchions. This reduction in critical load is due to the presence of an appreciable thrust in the cross-beam of the frame when vertical loads are applied to the cross-beam. The reduction in critical load noted by Chwalla is small (of the order of 3%) for the particular cases studied.

A characteristic feature of portal frames observed by Chwalla is that lateral buckling loads may be influenced by the presence of initial bending effects. In the paper a simple model portal frame is introduced and an exact elastic stability analysis shows that symmetric buckling loads are affected appreciably by initial bending effects. An exact treatment is also given for small antisymmetric displacements. The buckling loads are influenced again by initial bending effects, although not so seriously. For antisymmetric distortions the behaviour is reasonably linear until the onset of buckling. *R. Gran Olsson.*

Grilickiĭ, D. V. Distribution of stresses in an infinite anisotropic plate with elliptic opening as it is affected by the position of the point of application of the force and the momentum. Akad. Nauk Ukrain. RSR. Prikl. Meh. 2 (1956), 159–166. (Ukrainian. Russian summary)

Barenblatt, G. I. On certain problems of the theory of elasticity that arise in the investigation of the mechanism of hydraulic rupture of an oil-bearing layer. Prikl. Mat. Meh. 20 (1956), 475–486. (Russian)

Paslay, P. R.; und Slibar, A. Die Fliessbedingung und das Verformungsgesetz viskoser plastischer Stoffe. Österreich. Ing.-Arch. 10 (1956), 328–344.

Green, A. E. Hypo-elasticity and plasticity. II. J. Rational Mech. Anal. 5 (1956), 725–734.

[For part I see Proc. Roy. Soc. London. Ser. A. 234 (1956), 46–59; MR 17, 801.] The equations of hypo-elasticity are reformulated in terms of deviators and of Oldroyd's definition of stress rate. The author uses these to derive equations suitable for plastic flow of metals. He finds the most general equations such that the rate of change of the stress deviator is independent of the mean pressure of the dilatation, and he finds general conditions that two relations of this type, one for loading and one for unloading, be consistent. He considers special cases and finds that the equations of the usual theories of plasticity, suitably corrected so as to be meaningful for large strains, are included either by specialization or by a limiting process. *C. A. Truesdell* (Bologna).

Grigoryan, S. S. On formulation of dynamical problems for ideal plastic media. Prikl. Mat. Meh. 19 (1955), 725–733. (Russian)

A fluid medium is considered in which the density changes discontinuously at a certain pressure but is otherwise constant. For this medium, analysis is given of plane, cylindrical and spherical motion involving shock conditions. *H. G. Hopkins* (Sevenoaks).

Freiberger, W. Elastic-plastic torsion of circular ring sectors. Quart. Appl. Math. 14 (1956), 259–265.

The stress distribution and velocity field including warping in a circular ring sector with circular cross section under uniform torsion is obtained under the assumption of fully developed perfect plasticity of the section by the method of characteristics usually applied in the analysis of plane problems of plasticity. An inverse method due to von Mises and Sokolovsky is used in the solution of the elastic-plastic problem resulting in a nearly circular interface between the two regions.

The paper is a welcome increase of the very small number of existing solutions of boundary-value problems in the theory of the perfect plastic body that are not devoid of physical meaning. *A. M. Freudenthal.*

See also: Imšeneckaya, p. 388; Blagoveščenskiĭ and Fil'čakov, p. 399; Payne, p. 402; Pines, p. 418; Kron, p. 418; Hamel, p. 436; Diederich, p. 439; Roglić, p. 441.

Fluid Mechanics, Acoustics

★ **Hamel, G. Mechanik der Kontinua. Herausgegeben von István Szabó.** B. G. Teubner Verlagsgesellschaft, Stuttgart, 1956. 210 pp. DM 29.70.

This posthumous publication of G. Hamel's lectures contains several unusual features. Although the title of the book is Mechanics of Continua, it deals mainly with fluid mechanics. Only in the last part of the volume, "On general deformable systems" which occupies about one-tenth of the size of the book, is solid mechanics discussed; this part also includes some topics in plasticity and concludes with some remarks on rheology.

The part dealing with fluid mechanics is naturally divided into two parts: one on ideal fluids, one on viscous fluids. It is amazing how much the author has been

able to present in this small volume. It includes, for example, in the first part, the theory of waves of finite amplitudes, following Riemann and others, motion with free surfaces, vortex motion, hodograph method in compressible flow. In the part on viscous fluids, the distinguishing feature is the inclusion of some of the author's own contributions, such as the spiral motion and the "Streifenmethode" in the theory of turbulence. Such standard material as the Stokes theory and boundary layer theory are of course included.

Mathematical aspects of theory are dealt in great detail. The reader needs, however, only a knowledge of "Advanced Calculus" to follow the treatment. The presentation is accurate and clear as is befitting an original worker and experienced teacher. *C. C. Lin.*

★**Pai, Shih-I. Viscous flow theory. I. Laminar flow.** D. Van Nostrand Co., Inc., Princeton, N. J.-Toronto-New York-London, 1956. xvi+384 pp. $7.75.

The nature of this book is described as "A thorough study of laminar flow of viscous compressible fluids with special attention to aerodynamics and other engineering applications. (It is) written for research engineers and advanced students." It is the first of two volumes; the second volume — dealing with turbulent flow — will be published at a later date by the same publisher.

The fundamental equations are treated in the first three chapters: I. Physical Properties of Gases and Simple Kinetic Theory of Gases, II. Generalization of Laws of Friction and of Heat Conduction, III. Fundamental Equations of Fluid Dynamics of Viscous Compressible Fluids. Chapter IV gives examples of exact solutions of Navier-Stokes equations, including some cases in a compressible gas. After a brief chapter (Chapter V) on the law of similitude, some of the general mathematical properties of the Navier-Stokes equations are discussed in Chapter VI. Especially, the limiting cases of small and large Reynolds numbers are commented upon. This opens the way for one chapter (VII) on the theory of very slow motions and six chapters (VIII to XIII) on boundary layer theory. There is a final chapter on linearized theory of viscous compressible fluid.

The six chapters boundary layer theory occupy more than one-half of the size of the book. There is also some material on boundary layer theory in the final chapter. The following chapter headings will give an idea of the extensive nature of the treatment: VIII. The Boundary Layer Equations, IX .Exact solutions of Two-dimensional Boundary Layer Equations of Steady Flow, X. Approximate solutions of two-dimensional Boundary Layer Equations of Steady Flow, XI. Axially symmetrical and Three-dimensional Boundary Layer Flows, XII. Unsteady Flows in Boundary Layer, XIII. Boundary Layer Flows with Suction and with Injection. The latter half of Chapter XII is devoted to the theory of stability of the laminar boundary layer. It also includes a brief digression to other problems of instability of laminar flow.

The book contains an extensive compilation of information and references. It should be found helpful by all workers interested in the dynamics of viscous fluids.

C. C. Lin (Cambridge, Mass.).

Bernstein, Barry. On the uniqueness of ideal gas flows with a straight streamline. Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 850–854.

For steady, plane adiabatic flow of an ideal gas, the author shows that the streamline pattern determines the flow to within a Munk-Prim substitution provided one of the streamlines is straight and, on it, the normal derivative of the curvature of the orthogonal trajectories is not identically zero. Included is a proof that the streamline pattern and local Mach number always determine the flow to within a Munk-Prim substitution.

J. L. Ericksen (Washington, D.C.).

Lambin, N. V. On singular points of sigma-monogenic functions connected with an axially symmetric magnetic problem. Belorussk. Gos. Univ. Uč. Zap. Ser. Fiz.-Mat. **19** (1954), 27–31. (Russian)

Lambin, N. V. Reduction of an axially symmetric magnetic problem to finding Σ-monogenic functions. Belorussk. Goz. Univ. Uč. Zap. Ser. Fiz.-Mat. **15** (1953), 14–17. (Russian)

The equations considered are those of a rotationally symmetric incompressible flow in three dimensions. The author notes that these equations define sigma-monogenic functions and discusses two singular solutions.

L. Bers (New York, N.Y.).

Lighthill, M. J. Drift. J. Fluid Mech. **1** (1956), 31–53.

The word "drift" was used by C. G. Darwin [Proc. Cambridge Philos. Soc. **49** (1953), 342–354; MR **14**, 1027] to denote the deformation of material surfaces in fluid flows. The present study treats weakly-sheared flow past an obstacle, i.e., flow in which the parallel-stream velocity far upstream is nearly constant. In particular, results are worked out in detail for flow in which the shear in the parallel stream is nearly constant as well. The deformation of the vortex lines by the primary flow (the irrotational flow about the same obstacle with uniform stream velocity) is calculated, and thence the secondary flow field. Results are given without proof for the general case of weakly-sheared flow.

Returning to the uniform-shear case, the author has specialized it further to plane primary flows. For this subcase he is able to calculate the secondary flow field explicitly in terms of the drift function of the primary flow. Some analogous simplifications are shown for axisymmetric primary flow. The drift function for a sphere is studied and then applied to determine the vorticity field for this obstacle. One result is the secondary trailing vorticity (the streamwise component of vorticity far downstream), which can be compared with results of Hawthorne and Martin [Proc. Roy. Soc. London. Ser. A. **232** (1955), 184–195]. Agreement seems to be satisfactory. *W. R. Sears* (Ithaca, N.Y.).

Francis, J. R. D. The speed of drifting bodies in a stream. J. Fluid Mech. **1** (1956), 517–520.

"The speed of free floating bodies on the surface of a water stream in a sloping channel has been found to be sensibly the same as the mean speed of the layer of water in which the bodies are floating, contrary to some recorded opinions." (Author's summary.) One of these opinions is that of Brandtl, L. [Essentials of Fluid Dynamics, 1952; Blackie, London.]

Dyrbye, Claës. Détermination du point de déferlement pour une onde progressive en profondeur décroissante. Designation of the breaker-point for a progressive wave on decreasing depth. Houille Blanche **11** (1956), 415–418, discussion 347. (French and English versions)

Fassò, Costantino. Avviamento del moto di una corrente liquida in un tubo di sezione costante. Influenza delle resistenze. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. **90** (1956), 305–342.

Bazer, J.; and Karp, S. N. On a steady-state potential flow through a conical pipe with a circular aperture. J. Rational Mech. Anal. **5** (1956), 277–322.

Die betrachtete konische Röhre C entsteht aus einer rotationssymmetrischen Kegelfläche mit der Spitze im Koordinatenursprung und der $+x$-Achse als Symmetrieachse, indem man am Anfang die Punkte mit $r<b$ (r Abstand vom Ursprung, b gegeben) abschneidet. Gesucht wird das Geschwindigkeitspotential für die axialsymmetrische stationäre wirbelfreie Strömung einer inkompressiblen reibungsfreien Flüssigkeit, die, aus dem Unendlichen kommend, die Röhre durchfließt und durch das kreisförmige Ende verläßt. Neben den üblichen Bedingungen für das Randverhalten des Geschwindigkeitspotentials U und seiner Ableitungen auf C wird U für $r\to\infty$ innerhalb der Röhre vorgegeben und ferner verlangt, daß in der Nähe der kreisförmigen Kante der Ausflußöffnung sich die Geschwindigkeit wie $\delta^{-\frac{1}{2}}$ (δ Abstand von der Kante) verhält. Die Lösung wird als Integraldarstellung und als Reihenentwicklung erhalten; die Unität wird nachgewiesen. *K. Maruhn* (Dresden).

Polyahov, N. N. On induced forces in unsteady motion of a wing profile. Vestnik Leningrad. Univ. **11** (1956), no. 7, 87–93. (Russian)

Let x, y coordinate axes be rigidly attached to a thick airfoil k which moves through an incompressible fluid with instantaneous translatory velocity $u_c(t)$, $v_c(t)$ and angular velocity $\omega(t)$. The velocity potential function

$$\Phi(x, y, t)= u_c\Phi_1(x, y)+v_c\Phi_2(x, y)+\omega\Phi_3(x, y) +(\Gamma/2\pi)\Phi_4(x, y)+\Phi_s(x, y, t),$$

where Φ_s is the velocity potential due to the trailing vortex sheet CD and $\Gamma(t)$ is the circulation about k. The force on k is

$$R^*=-i\rho\Gamma(u_c-iv_c)-\rho\omega\int_k \bar{z}d\Phi +\tfrac{1}{2}i\rho\int_k(u-iv)^2dz+i\rho\int_k(\partial\Phi/\partial t)d\bar{z},$$

where $z=x+iy$ and $\bar{z}=x-iy$. By considering a region bounded externally by a circle of large radius and internally by k and CD the author shows that

$$I_1=\int_k(u-iv)^2dz= -\int_{CD}[(u-iv)^2{}_B-(u-iv)^2{}_H]dz=-2\int_{CD}\gamma(u^*-iv^*)ds,$$

where subscripts $B(H)$ denote upper (lower) side of CD, γ is the intensity and u^*, v^* is the velocity of the vortex at the point of arc length s on CD. If the bound vortices are on the surface of k with intensity γ_{II} at the point of arc length s_1 on k the author interprets $\frac{1}{2}i\rho I_1$ as an induced force

$$\mathbf{R}_1^*=\frac{1}{2\pi}\int_{CD}\int_k[\boldsymbol{\gamma}\times(\boldsymbol{\gamma}_{\text{II}}\times\mathbf{r}_1]r_1^{-3}ds_1\,ds,$$

where the vectors $\boldsymbol{\gamma}$ and $\boldsymbol{\gamma}_{\text{II}}$ of magnitudes γ and γ_{II} are normal to the z-plane and $\mathbf{r}_1$ is the vector $z(s)-z(s_1)$.

J. H. Giese.

Weissinger, Johannes. Neuere Entwicklungen in der Tragflügel-theorie bei inkompressibler Strömung. Z. Flugwiss. **4** (1956), 225–236.

Dieser Vortragsbericht behandelt unter einheitlichem Gesichtspunkt neuere und neueste theoretische Untersuchungen zur Tragflügeltheorie, soweit es sich um einen einzelnen Flügel bei stationärer Strömung in einer reibungsfreien inkompressiblen, den ganzen Raum erfüllenden Flüssigkeit handelt; betrachtet werden ferner nur Theorien, bei denen das Strömungsfeld durch diskret oder schichtweise angeordnete Wirbel erzeugt wird, wobei wesentlich (bezüglich des Anstellwinkels) lineare und nichtlineare Theorien zu unterscheiden sind. Die ersteren, die naturgemäß in dem Bericht den weitaus größten Raum einnehmen, lassen sich auffassen als Näherungen der Theorie der tragenden Fläche. Hierher gehören die Arbeiten über das ebene Problem, die Streifenmethode, die Prandtlsche Traglinientheorie, Jones' Theorie für Flügel kleiner Streckung und gewisse Verschräfungen der Theorien von Prandtl und Jones. Bei der "erweiterten Traglinientheorie" z.B., deren Ausbau man vor allem dem Verf. verdankt, wird die tragende Fläche durch einen diskreten tragenden Wirbel in der $l/4$-Linie (l Flügeltiefe) ersetzt und die Abwindbedingung auf der $3l/4$-Linie erfüllt. Der Fortschritt liegt in der Anwendbarkeit auch auf kleinere Flügelstreckungen, ferner auf Flügel mit beliebigem Grundriß, z.B. auf Pfeilflügel. — Verf. berichtet sodann über verschiedene Behandlungsmethoden für die eigentliche Tragflächengleichung (Multhopp, Truckenbrodt), über Untersuchungen zum schiebenden und gepfeilten Flügel (Weissinger, Legras, Jaeckel) und über seine kürzlich entwickelte Theorie des Ringflügels. Den Abschluß bilden Hinweise auf das viel verwendete Prinzip des "reverse-flow" und auf die Gaußschen Quadraturformeln, eine Zusammenstellung der wenigen nichtlinearen Theorien (z.B. Flax u. Lawrence, Brown u. Michael, Jones) und eine überaus reichhaltige Zusammenstellung des einschlägigen Schrifttums. *K. Maruhn.*

Pyhteev, G. N. Exact solution of the problem of Kirchhoff's flow with separation for one family of curves. Dokl. Akad. Nauk SSSR (N.S.) **108** (1956), 34–37. (Russian)

In der vorliegenden Arbeit wird eine zweiparametrige Kurvenschar angegeben, für die bei einer reibungsfreien inkompressiblen Flüssigkeit konstanter Dichte das Strömungsfeld nebst den freien Stromlinien bei Abreißen durch strenge, für die Rechnung geeignete Formeln erhalten werden kann. Für einen speziellen Wert des einen Parameters werden die Betrachtungen weiter verfolgt und numerische Ergebnisse, insbesondere die Widerstandskoeffizienten, erhalten. *K. Maruhn* (Dresden).

Gotusso, Guido. La teoria dello strato limite nel caso tridimensionale. Aerotecnica **36** (1956), 95–100.

Facendo uso della metrica della superficie e dello spazio contiguo e facendo una prima valutazione degli ordini di grandezza dei termini che campaiono nelle equazioni dei fluidi viscosi, si ottengono, in modo del tutto analogo al caso piano, delle equazioni ridotte, in cui figurano derivate tensoriali. Poichè la scelta delle coordinate sulla superficie è largamente arbitraria, si scelgono come linee coordinate su di essa le linee di corrente e le linee equipotenziali corrispondenti all corrente linera (non viscosa). Questa scelta comparta notevoli semplificazioni dei calcoli. Lo sviluppo successivo delle derivate tensoriali permette allora di operare una

seconda valutazione di ordini di grandezza, che porta finalmente a formule ridotte direttamente impiegabili in qualsiasi caso. *Riassunto dell'autore.*

Nikitin, A. K. On the motion of viscous fluid between pin and bearing. Inžen. Sb. **23** (1956), 173–185. (Russian)

Žukovskiĭ et Čaplygin [Trudy Otd. Fiz. Nauk Obšč. Lyubitel. Estest. **13** (1906), 24–33=Žukovskiĭ Oeuvres collectives, t. 3, Gostehizdat, Moscow-Leningrad, 1949, pp. 133–151] ont donné une solution de ce problème en considérant le cas plan et en négligeant les forces d'inertie. I. M. Fišman [Prikl. Mat. Meh. **14** (1950), 593–610; MR **12**, 763] a donné une solution en introduisant en deuxième approximation les forces d'inertie.

L'auteur reprend le problème dans le plan en coordonnées polaires en introduisant dans les équations les forces d'inertie. En développant la fonction du courant suivant les puissances croissantes d'un petit paramètre (l'excentricité relative) l'auteur arrive à déterminer cette fonction avec une approximation aux termes du 3eme ordre près en utilisant les fonctons de Bessel.

M. Kiveliovitch (Paris).

Muggia, Aldo. Velocità di evaporazione e coefficiente di resistenza per una goccia liquida in corrente gassosa. Aerotecnica **36** (1956), 127–131.

Struminskiĭ, V. V. Three-dimensional boundary layer on an arbitrary surface. Dokl. Akad. Nauk SSSR (N.S.) **108** (1956), 595–598. (Russian)

Die Gleichungen für die dreidimensionale Grenzschicht werden für Re→∞ (Re· Reynoldssche Zahl) aus den Navier-Stokesschen Gleichungen hergeleitet. Hierbei wird angenommen, daß die Kontur des umströmten Körpers stetig differenzierbare Hauptkrümmungen hat und die Geschwindigkeit in der Nachbarschaft beim Grenzübergang nebst den Ableitungen erster und zweiter Ordnung endlich bleibt. Es werden Spezialfälle hervorgehoben, in denen Analogie zu den Gleichungen der dreidimensionalen Grenzschicht für die ebene Platte besteht.

K. Maruhn (Dresden).

Napolitano, Luigi G. Soluzioni esatte per lo strato limite laminare in presenza di gradiente assiale di pressione e di iniezione di fluido. Aerotecnica **36** (1956), 132–141.

Weber, H. E. The boundary layer inside a conical surface due to swirl. J. Appl. Mech. **23** (1956), 587–592.

Betchov, R. An inequality concerning the production of vorticity in isotropic turbulence. J. Fluid Mech. **1** (1956), 497–504.

The author gives a direct derivation of relations connecting the moments of a velocity derivative in terms of the moments of the principal rates of strain for the case of isotropic turbulence. He then derives an inequality $|S| \leq 2\gamma^{1/2}/\sqrt{21}$ connecting the flatness factor γ and the skewness factor S for the first derivative. Comparison with experimental values of S and γ gives certain suggestions regarding the predominant types of distortion occurring in homogeneous isotropic turbulence.

C. C. Lin (Cambridge, Mass.).

Phillips, O. M. On the aerodynamic surface sound from a plane turbulent boundary layer. Proc. Roy. Soc. London. Ser. A. **234** (1956), 327–335.

It is shown that the aerodynamic surface sound generated by the interaction of turbulent motion with an infinite plane surface vanishes if the turbulence is statistically homogeneous in planes parallel to the bounding surface. The proof of this is based on the theorem:

$$(*) \qquad \int \left\langle \frac{\partial F_i}{\partial t} g' \right\rangle_{\text{Av}} d\Sigma = 0,$$

where F_i is the force per unit area acting upon the fluid at a point η of the boundary Σ, $g(\mathbf{x}') \equiv g'$ is some dynamical property of the fluid at a fixed point $\mathbf{x}'$, and the integration is effected over an infinite surface on which the normal component of the velocity is zero, or over the surface of a closed rigid body. In case the turbulence is homogeneous in planes normal to the x_1-axis, the author shows that one can deduce from (*) that

$$\iint \langle v_i(\mathbf{x}) v_j(\mathbf{x}') \rangle d\mathbf{x} dx_1' = 0 \quad (\mathbf{x}' = \mathbf{x} + \mathbf{r}).$$

From this last result the further relations

$$\iint \left\langle \frac{\partial^2 v_i}{\partial t^2} \frac{\partial^2 v_j'}{\partial t^2} \right\rangle_{\text{Av}} dx dx_1' = 0,$$

$$\int \left\langle \frac{\partial F_i}{\partial t} \frac{\partial F_j'}{\partial t} \right\rangle_{\text{Av}} d\Sigma = 0$$

are derived. Since the acoustic intensity generated is proportional to this last quantity [cf. Curle, same Proc. **231** (1955), 505–514; MR **17**, 681] the result stated at the beginning (regarding this vanishing) follows.

S. Chandrasekhar (Williams Bay, Wis.).

Diederich, Franklin W. The dynamic response of a large airplane to continuous random atmospheric disturbances. J. Aero. Sci. **23** (1956), 917–930.

The statistical approach to the gust-load problem, which consists in considering flight through turbulent air to be a stationary random process, is extended by including the effect of lateral variations of the instantaneous gust intensity on the aerodynamic forces and on the resultant motions and stresses of rigid and flexible airplanes. By means of some calculations of normal and rolling accelerations, as well as of the root bending moment, it is shown that these effects may be significant for large airplanes. (From the author's summary.)

R. C. DiPrima (Los Angeles, Calif.).

Slobodov, B. Ya. Consideration of turbulent exchange in the problem of distribution of pressure and wind in the atmosphere. Izv. Akad. Nauk SSSR. Ser. Geofiz. 1956, 1001–1004. (Russian)

On connait d'après les recherches de E. Blinova [C. R. (Dokl.) Acad. Sci. URSS (N.S.) **39** (1943), 257–260; MR **5**, 194] les expressions de la pression moyenne et du vent moyen si l'on se donne d'avance la distribution des températures dans l'atmosphère. Dans ces recherches on néglige la turbulence et on considère l'atmosphère comme un fluide idéal barocline. D'autre part A. Mhitaryan [Izv. Akad. Nauk SSSR Ser. Geofiz. **1955**, 80–83] a donné une méthode pour calculer l'échange turbulent vertical.

En utilisant ces recherches l'auteur généralise le problème en introduisant l'influence de l'échange horizontal macro-turbulent.

En utilisant 1) — l'équation vectorielle du tourbillon de Fridman, 2) — l'équation du mouvement d'Euler le long de la latitude ainsi que 3) — l'équation de continuité, l'auteur arrive, après un certain nombre de transformations, et en linéarisant les équations, à un résultat dont les cas limites sont les équations de Blinova et Mhitaryan.
M. Kiveliovitch (Paris).

Mattioli, Ennio. Ricerche teoriche e sperimentali sulla turbolenza di parete. Aerotecnica **36** (1956), 112–126.

Corcos, G. M.; and Liepmann, H. W. On the contribution of turbulent boundary layers to the noise inside a fuselage. NACA Tech. Memo. no. **1420** (1956), ii+43 pp.

Laitone, E. V. On equations of motion for a compressible viscous gas. J. Aero. Sci. **23** (1956), 846–854.

This is an exposition presenting to engineers the author's interpretation of recent mathematical and physical investigations seeking to determine equations of motion for moderately rarefied gases both by continuum and kinetic theory approaches. Among other things, the author discusses the origin of the special forms produced by the Chapman-Enskog expansion, the effect of particular molecular models on the general equations, the effect of vorticity terms and vortex friction, and the insensitivity of the Navier-Stokes approximation to the physical model.
C. A. Truesdell (Bologna).

Schmitz, H. P. Ein neuer Beweis des Theorems von H. Ertel über asynchron-periodische Wirbelbewegungen kompressibler Flüssigkeiten. Acta Hydrophys. **3**(1956), 147–155.

H. Ertel has recently shown relations between circulation, energy and period of rotation of asynchronous-periodic vortex motions of compressible fluids through the use of the equations of motion in the Lagrangian form [Miscellanea Acad. Berolinensia, v. 1, Akademie-Verlag, Berlin, 1950, pp. 62–68; MR **14**, 508]. The same relations are here proved through the use of the equations of motion in the Eulerian form.
Hirsh Cohen.

★ **Станюкович, К. П. [Stanyukovič, K. P.] Неустановившиеся движения сплошной среды. [Unsteady motion of a continuous medium.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1955. 804 pp. 26.35 rubles.

This is an exhaustive treatise on unsteady motion of a compressible gas with particular application to the theory of explosions and to internal ballistics. Much material is assembled which has previously been widely scattered in research publications and, in translated form, would be of great value to workers in the explosives field, since, at present, there is no equivalent book written in English.

The first two chapters deal with thermodynamical concepts and the fundamental equations of motion, including an account of the method of characteristics. Chapter III is concerned with self similar motions of media and contains much new material due to Sedov and his group. Chapter IV describes special solutions for isentropic flow corresponding to $\gamma=3$ and other simple gas laws. Chapter V treats one dimensional isentropic motion, especially problems of interaction of rarefaction waves. The three following chapters deal very thoroughly with the properties of shock waves and detonation waves. Chapter IX gives a general account of the propagation of shock waves in unsteady flow. Chapter X is concerned with the motion of detonation and blast waves with cylindrical or spherical symmetry; special attention is given to solutions based on similarity hypotheses. Chapter XI concerns unsteady motion in dense media with emphasis on cavitation phenomena; a section on river waves is also included. Chapter XII contains a complete discussion of Lagrange's Ballistic Problem. The remaining chapters deal with effects of gravity and relativity in gas dynamics and are clearly intended for application to astrophysical problems.

The book sets out a large number of papers, some nearly 20 years old, which have evidently been classified previously. It is extremely long, partly due to the inclusion of too much detail in mathematical development, and partly due to repetition of results. However the style is very clear and, since most leading problems of current interest in unsteady gas dynamics are discussed at length, the book can be recommended as a useful work of reference.
M. Holt (Providence, R.I.).

Nužin, S. G. On the flow of a gas at high subsonic velocities about grids. Kazan. Aviac. Inst. Trudy **29** (1955), 3–7. (Russian)

In a previous paper [Prikl. Mat. Meh. **10** (1946), 657–666; MR **8**, 417], the author presented an approximate method for computing subsonic flows past a profile, the heuristic idea being that the mass flow vector of a compressible flow can be interpreted as the velocity vector of a certain rotational flow of an incompressible fluid. In the present paper this method is applied to the flow about a grid of profiles.
L. Bers (New York, N.Y.).

Dressler, Robert F. Entropy changes in rarefaction waves. J. Res. Nat. Bur. Standards **57** (1956), 265–271.

L'auteur étudie les mouvements rectilignes non stationnaires d'un fluide compressible, le frottement du fluide sur les parois est pris en considération. Dans une première partie son action est rameéne à celle d'une force opposée à la vitesse et proportionnelle au carré de la vitesse; dans une seconde partie cette action est supposée se traduire en autre par une dissipation d'énergie sous forme de chaleur et par une variation d'entropie. Les équations obtenues ainsi sont étudiées au moyen d'approximations successives, la première approximation correspondant aux ondes simples concourantes du cas isentropique classique.
H. Cabannes (Québec).

Korst, H. H. A theory for base pressures in transonic and supersonic flow. J. Appl. Mech. **23** (1956), 593–600.

Guiraud, Jean-Pierre. Forces aérodynamiques non stationnaires sur une aile mince de très faible allongement en déformation. C. R. Acad. Sci. Paris **243** (1956), 1278–1281.

The work initiated by Merbt and Landahl [Roy. Inst. Tech., Div. Aero., KTH Aero. TN **30** (1953)], who calculated the forces on a slender, rigid, oscillating aerofoil in a compressible stream using elliptic coordinates, is extended by using a Green's function technique to cover the case of a flexible aerofoil. The author shows how, given the mode of deformation (which must be small since linearised theory is used throughout), the forces acting may be calculated. The results are given as double and triple summations which involve Mathieu functions.
G. N. Lance (Southampton).

Woods, L. C. Theory of aerofoil spoilers. Aero. Res. Council, Rep. and Memo, no. **2969** (1953), 21 pp. (1956).

A mathematical theory of aerofoil spoilers in two dimensional subsonic flow is presented. Equations are given for load distributions, lift, drag, moments and hinge moments produced by spoiler-flap combinations. The theory is developed for a spoiler in a general position but the trailing-edge spoiler receives special attention. For this important case the theory gives good agreement with experiment, but in the more general case, because of uncertainty about the pressure distribution on the aerofoil to the rear of the spoiler, the agreement is not as good. (Author's summary.) *R. M. Morris.*

Pivko, Svetopolk. Zur Abschätzung der aerodynamische Eigenschaften dünner kreiszylindrischer, schrägangeströmter Ringflügel. Z. Angew. Math. Mech. **36** (1956), 306–307.

Kučerenko, È. G. A special case of movement of ground waters. Tomskiĭ Gos. Univ. Uč. Zap. Mat. Meh. **25** (1955), 113–114. (Russian)

L. A. Galin [Prikl. Mat. Meh. **15** (1951), 655–678; MR 13, 397] illustrated a method of successive approximations for the solution of a planar problem of non-stabilized flow of ground water. The present brief note gives the exact solution of that problem. The author obtains this solution by determining the coefficients of a certain analytic function (that is the solution of a more general problem) so as to fit the particular problem at hand.
H. P. Thielman (Ames, Ia.).

Četaev, D. N. On the influence of the velocity of subsonic flow on the radiation impedance of a piston with infinite baffle. Akust. Ž. **2** (1956), 302–309. (Russian)

The medium flows in the half-space $z>0$ with velocity u parallel to the y-axis, where $u<c$ (the velocity of sound). A piston oscillates in a cavity in the rigid screen $z=0$. The problem is to evaluate a quadruple integral giving the impedance. After reproducing his earlier manipulations [Dokl. Akad. Nauk SSSR (N.S.) **90** (1953), 355–358; MR **15**, 178] the author gives a further simplification for the case of a rectangular piston. Also given are tables for the cases $u/c=0$, 0.2, 0.4, 0.6 and 0.8 and varying sizes of square piston. *F. V. Atkinson* (Canberra).

Barducci, Italo. Costanti acustiche dei tubi di piccola sezione. Alta Frequenza **25** (1956), 355–377.

See also: Yeh, Martinek, and Ludford, p. 388; Sagomonyan, p. 400; Bergman, p. 401; Payne, p. 402; Paslay and Slibar, p. 436; Grigoryan, p. 436; Tesson, p. 442; Message, p. 448; Andreoletti, p. 448.

Optics, Electromagnetic Theory, Circuits

de Belatini, Paul. Multipotentials of multipoles. Bull. Tech. Univ. Istanbul **8** (1955), 57–74. (Turkish summary)

In einer früheren Arbeit [derselbe Bull. **5** (1952), 27–110] erläuterte der Verfasser am Beispiel magnetischer Felder, daß zu allen physikalischen Phänomenen und Konfigurationen gewisse physikalische Größen gehören, mit deren Hilfe die mathematische Beschreibung am einfachsten und die entsprechenden Felder am symmetrischsten sind. In der vorliegenden Arbeit wird dies Prinzip angewendet auf (elektrische oder magnetische) Felder von verschiedenen Multipolen (Monopolen, Dipolen, Quadripolen und Oktopolen). Diese Felder lassen sich meist mit Hilfe eines skalaren Potentials oder seines Gradienten beschreiben. Zu jedem Multipol gehört eine spezielle skalare oder vektorielle Potentialgröße mit Kugelsymmetrie. *K. Maruhn* (Dresden).

Hansen, Robert C. Electromagnetic field solutions for rotational coordinate systems. Canad. J. Phys. **34** (1956), 893–895.

In this paper the solution of electromagnetic problems in rotational systems is approached by the vector potential method. A solution is assumed in the form $E=\mathrm{curl}\, a_3f$, where E is the electric field strength, a_3 a unit vector in the angular ϕ direction, and f a scalar. Then: $\mathrm{curl}\,\mathrm{curl}\, a_3f=a_3k^2f$. By expansion of the left hand side in the rotational coordinates a differential equation for f is obtained, which may be simplified in the following 5 cases: circular cylindrical, spherical, prolate and oblate spheroidal and rotational parabolical. From the f-solution the field is derived and the physical interpretation is arrived at. *M. J. O. Strutt* (Zurich).

Roglić, Velimir. Anwendung der Saint-Venant-schen Bedingung in der Elektrodynamik. Glas Srpske Akad. Nauka 206. Od. Prirod.-Mat. Nauka (N.S.) **5** (1953), 39–42. (Serbo-Croatian. German summary)

The author wishes to prove, on a classical basis, that an electromagnetic field cannot in general be identified with the stresses in an elastic medium. Attempting such an identification for the special case of the electrostatic field of a point charge, and applying the compatibility conditions for the stresses of an elastic medium, he deduces that the Poisson ratio of the medium would have to have the (unacceptable) value unity. *F. V. Atkinson.*

Durand, Emile. Les densités singulières de l'électrostatique et de la magnétostatique. Ann. Sci. Ecole Norm. Sup. (3) **73** (1956), 75–91.

It is shown that the use of the expression $\Delta(1/r)$ may lead to more general and more satisfying formulations in many cases. Applications are given showing the advantage of using $\Delta(1/r)$ in volume, surface, and line integrals. A similar presentation is given for integrals involving $\Delta \log r$. The general theory is applied to singular densities of electrostatics and magnetostatics. The special cases considered in electrostatics include volume densities corresponding to point, linear, and surface source distributions for sources consisting of a single charge, a dipole, or a multipole. In the magnetostatic case volume densities corresponding to linear and surface currents are considered. *J. E. Rosenthal* (Passaic, N.J.).

Horváth, J. I. On the theory of the electromagnetic field in moving dielectrica. Bull. Acad. Polon. Sci. Cl. III. **4** (1956), 447–452.

The author sets up a variational principle for Maxwell's equations in the presence of matter using the polarization rather than the D and B vectors. He then shows that the conservation identities following from this variational principle involve the energy-momentum tensor as defined by Abraham. He also states that the energy-momentum tensor as defined by Minkowski also satisfies the conservation identities. From this he draws the conclusion

that the laws of conservation are not suitable to settle which energy-momentum tensor is the correct one, as supposedly suggested by others. [See Balazs, Phys. Rev. (2) **91** (1953), 408–411; MR **15**, 186; G. Györgyi, Acta. Phys. Akad. Sci. Hungar. **4** (1954), 121–131; MR **16**, 775.] {The author's conclusions seem to be based on a misconception. The papers quoted do not use only the conservation laws, but also the equations of motion for the dielectric. Thus in addition they require that the rate of change of the momentum of the dielectric exposed to the electromagnetic field should be equal to the force acting on it, the force being computed from the energy-momentum tensor. This condition is only satisfied by the Abraham tensor and not by the Minkowski one.}

N. L. Balazs (Chicago, Ill.).

Hurd, R. A. Radiation patterns of a dielectric-coated axially-slotted cylinder. Canad. J. Phys. **34** (1956), 638–642.

By the well-known Fourier-Lamé method the author finds a series representation for the far-zone radiation field of a perfectly conducting circular cylinder coated uniformly with a lossless dielectric and excited by an axial slot across which is applied an oscillating electric field. He evaluates the series numerically and compares with experimental results. *C. H. Papas.*

Bertein, François; et Chahid, Wassek. Sur la production d'ondes électromagnétiques lentes à l'aide de nappes de courant cylindriques. C. R. Acad. Sci. Paris **242** (1956), 2918–2920.

The authors examine the propagation of slow waves along a periodic structure consisting of spires of arbitrary shape wound on a circular cylinder. *C. H. Papas.*

Toraldo di Francia, Giuliano. Equazioni integrodifferenziali e principio di Babinet per gli schermi piani a conduttività unidirezionale. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **20** (1956), 476–480.

This paper deals with Babinet's principle in diffraction of electromagnetic waves by a thin plane screen with uni-directional conductivity. Such a screen can be realised approximately by fixing to a thin insulating sheet parallel perfectly conducting straight wires, the distance between adjacent wires being very small compared with the wave length.

It is shown that the screen complementary to a disc of uni-directional conductivity is an identical disc set into an infinite perfectly conducting plane screen.

E. T. Copson (St. Andrews).

Aymerich, Giuseppe. Sui modi liberi di un campo elettromagnetico sostenuto da un guscio elicoidale. Rend. Sem. Fac. Sci. Univ. Cagliari **26** (1956), 105–117.

The paper deals with the propagation of "monochromatic" electromagnetic waves along a wave guide of somewhat unusual construction. The outer conductor is a perfectly conducting straight metal tube; the inner conductor is a straight metal tube with helical conductivity.

The case when the cross-sections of the two conductors are concentric circles is discussed in detail.

E. T. Copson (St. Andrews).

Horvay, G.; Linkous, C.; and Born, J. S. Analysis of short thin axisymmetrical shells under axisymmetrical edge loading. J. Appl. Mech. **23** (1956), 68–72.

Formulas are given for stresses and deflections in axisymmetrical bending of thin shells of revolution under the assumption that at the two circular edges of the shell either stresses or displacements are prescribed. Use is made of known results of asymptotic integration methods in shell theory. The essential point of the present work is that the two circular edges are so close to each other that it is not possible to neglect the interaction of the two edge effects and it becomes important to arrange the calculations judiciously. *E. Reissner.*

Keller, Joseph B. Electrohydrodynamics. I. The equilibrium of a charged gas in a container. J. Rational Mech. Anal. **5** (1956), 715–724.

The author considers a uniformly charged gas in a cavity which is completely bounded by a perfectly conducting wall. Assuming the charged gas to be a macroscopic continuum in equilibrium, he proves that the mass density and the pressure are constant and maximum on the wall and approach finite upper limits as the total mass of the gasis increased indefinitely.

C. H. Papas (Pasadena, Calif.).

Deards, S. R. Notes on the theory of planar electric networks. Coll. Aero. Cranfield. Note no. **52** (1956), i+30 pp. (4 plates).

This is a concise formal exposition of the algebra of planar networks involving the relations between incidence and impedance-admittance matrices, in dual manner with respect to node and mesh representations. The formalism is extended to general independent circuital current, potential difference, and hybrid systems, again with the stress on strict duality. Examples are given. The irrelevance of the symmetry type to the power behavior of electrical networks is pointed out.

H. G. Baerwald (Cleveland, Ohio).

Zemanian, Armen H. Restrictions on the shape factors of the step response of positive real system functions. Proc. I. R. E. **44** (1956), 1160–1165.

Epstein, Bernard. Determination of coefficients of capacitance of regions bounded by collinear slits and of related regions. Quart. Appl. Math. **14** (1956), 125–132.

The author determines the coefficients of capacitance of a bi-filar cable by conformally mapping the cable domain into a slit domain. *C. H. Papas.*

See also: Yeh, Martinek, and Ludford, p. 388; Bolinder, p. 393; Lauwerier, p. 400; Park, p. 405; Cole, p. 430; Frisch and Wilets, p. 445; Elsasser, p. 448; Oblogina, p. 449.

Thermodynamics and Heat

Tesson, Fernand. Application du premier principe aux systèmes fluides limités par une surface de contrôle variable. C. R. Acad. Sci. Paris **243** (1956), 560–562.

Here are continued the investigations of the author about the systems, with a variable number of particles enclosed by a mobile surface, which are of interest for the problems of the self-propulsion. He has already published two notes about some kinematical and dynamical properties of such systems [same C. R. **240** (1955), 845–847, 1050–1052; MR **16**, 1168]. Here is given the application of the first thermodynamical law to such systems and

two relations are derived which ought to be used in a following publication for an extension of the definitions of the coefficient of efficiency for the propelled systems.
T. P. Anđelić (Belgrade).

Jarre, Gianni. Il raffreddamento evaporativo ad alta velocità. Aerotecnica 36 (1956), 101–111.

Kyame, Joseph John. Matrix representation of thermodynamic fundamentals. Amer. J. Phys. 25 (1957), 67–69.

The use of matrices for representing fundamental thermodynamic relations is demonstrated. Maxwell's relations and other thermodynamic derivatives are readily obtained by differentiation of the matrices defined. *From the introduction.*

See also: Yeh, Martineh, and Ludford, p. 388; Fulks, p. 398; Lauwerier, p. 400; Rubin and Shuler, p. 429; Hooton, p. 430; Pai, p. 437; Dressler, p. 440; Guiraud, p. 440; Tatarskiĭ, p. 448.

Quantum Mechanics

McWeeny, R. The density matrix in self-consistent field theory. I. Iterative construction of the density matrix. Proc. Roy. Soc. London. Ser. A. 235 (1956), 496–509.

The wave function of an electronic system is in the Hartree-Fock scheme approximated by a single Slater determinant built up from the actually occupied spin-orbitals. It has been pointed out by several authors that, in this scheme, the physical situation is entirely described by the first-order density matrix and that the knowledge of the individually occupied orbitals is not required. The goal of the self-consistent-field procedure is usually to determine these orbitals or Hartree-Fock functions, but McWeeny describes instead a similar method for the direct evaluation of the density matrix as being the fundamental quantity. An application to the ground state of the beryllium atom shows good result.

In an appendix is described a method of "steepest descent" for solving eigenvalue problems and an iterative method for constructing idempotent matrices.
P.-O. Löwdin (Uppsala).

Sokolov, A. A.; and Kerimov, B. K. On the scattering of spin-less particles taking into account damping. Dokl. Akad. Nauk SSSR (N.S.) 108 (1956), 611–614. (Russian)

The scattering of a spinless particle by a static short-range potential is described by means of the Klein-Gordon equation. An approximate solution of the equation is given which reduces at $t=0$ to a state of definite momentum. By including the effects of damping, it is arranged that the approximate solution yields strict conservation of probability. Formulae are given for the differential and total scattering cross sections which follow from the wave function. Approximate expressions for the phase shifts and total cross section for small momenta and large angular momentum are worked out in the special case of the Yukawa potential. *A. S. Wightman.*

Cirelli, R.; e Pusterla, M. Estensione del metodo parametrico di Davison al caso di potenziali cinetici. Nuovo Cimento (10) 4 (1956), 150–153.

The authors compute the propagation function of Schrödinger electron in an external homogeneous magnetic field both directly and by the Feynman path integral method discussed by Davison [Proc. Roy. Soc. London. Ser. A. 225 (1954), 252–263; MR 16, 319].
A. S. Wightman (Princeton, N.J.).

Costa de Beauregard, Olivier. Le réalisme de l'espace-temps: sur deux problèmes d'interprétation en mécanique ondulatoire. C. R. Acad. Sci. Paris 243 (1956), 1838–1840.

Dalgarno, A.; and Lewis, J. T. The equivalence of variational and perturbation calculations of small disturbances. Proc. Phys. Soc. Sect. A. 69 (1956), 628–630.

The authors display explicitly the equivalence of three different methods of calculating the perturbed energy of a quantum mechanical system, the Slater-Kirkwood method, the method of Schrödinger perturbation theory, and a variational method. *A. S. Wightman.*

Skinner, R. A quantum mechanical description of collective motions. Canad. J. Phys. 34 (1956), 901–913.

A rigorous and general formalism is presented for the quantum mechanical description of the collective motions of a system containing a finite number of particles. The dynamical equations are expressed in terms of collective and "internal" coordinates. The internal coordinates are symmetrical in the particles, and correspond to the laboratory coordinates with the collective motions subtracted off. The formalism is applied to center of mass motion. [From the author's abstract].
T. E. Hull (Vancouver, B.C.).

Castoldi, Luigi. Osservabili e operatori condizionati nei fondamenti della meccanica quantica. Rend. Sem. Fac. Sci. Univ. Cagliari 26 (1956), 83–89.

★**Chintschin, A. J. Mathematische Grundlagen der Quantenstatistik.** Akademie-Verlag, Berlin, 1956. ii+200 pp. DM 21.00.

A translation by Ernst Wilde of the book reviewed in MR 13, 894.

★**Швингер, Ю. [Švinger, Yu. (Schwinger, J.)] Теория квантованных полей. The theory of quantized fields.]** Izdat. Inostran. Lit., Moscow, 1956. 252 pp. 10.25 rubles.

A translation by N. P. Klepikov and L. F. Lapidus from the English of the work reviewed in MR 15, 1010.

Stepanov, B. M. Non-relativistic regularization of the S-matrix. Dokl. Akad. Nauk SSSR (N.S.) 108 (1956), 1045–1047. (Russian)

The author describes briefly a regularization procedure for quantum field theories which is covariant only in the physical limit but preserves the hermiticity of the hamiltonian even before passage to the limit. Considering the special case of electrodynamics he gives explicitly the counter-terms which are equivalent to his regularization method. The final results for the S-matrix are the same as in other renormalization methods. Finally, he remarks that the method yields $-\infty$ for the renormalization constant Z_3 in the physical limit, just as found by Bogolyubov and Širkov [same Dokl. (N.S.) 105 (1955), 685–688; MR 17, 1033]. *A. S. Wightman.*

Kastler, Daniel. Sur l'espace de la théorie quantique des champs. Ann. Univ. Sarav. **4** (1955), 206–237 (1956).

This is an expository article devoted to the mathematics of second quantization. The author treats successively Hilbert spaces in general, the tensorial products of Hilbert spaces, the tensor algebra constructed on Hilbert space, the symmetrized and antisymmetrized tensor algebras and the operations corresponding to the creation and annihilation operators of second quantization.

L. Van Hove (Utrecht).

Solov'ev, V. G. On a certain model in the quantum theory of fields. Dokl. Akad. Nauk SSSR (N.S.) **108** (1956), 1041–1044. (Russian)

The author proposes a model of a quantum field theory of interaction of a somewhat different type from those studied previously. [A model is by definition a theory which satisfies some but not all of the requirements one would impose on a relativistic quantum theory of fields. See, e.g., T. D. Lee, Phys. Rev. (2) **95** (1954), 1329–1334; MR **16**, 317, and G. Källén and W. Pauli, Danske Vid. Selsk. Mat. Fys. Medd. **30** (1955), no. 7; MR **17**, 927.] He writes down a history-integral for the nucleon Green function in neutral pseudo-scalar meson theory with pseudo-scalar coupling. He then calculates it approximately using the double limit technique of Abrikosov and Halatnikov [Dokl. Akad. Nauk SSSR (N.S.) **103** (1955), 993–996; MR **17**, 565] and the assumption that the limiting meson momentum is zero and the limiting nucleon momentum infinity. The model referred to in the title is obtained by regarding this approximation procedure as defining a theory in which the procedure is exact. After renormalization, the nucleon Green's function of the model satisfies all the general requirements imposed by positive definiteness of the scalar product and relativistic invariance. *A. S. Wightman* (Princeton, N.J.).

Pekar, S. I. On the existence of stationary quantum states of point nucleons interacting with a meson field. Ž. Eksper. Teoret. Fiz. **29** (1955), 599–604. (Russian)

The question of the existence of stationary states of a system consisting of point nucleons interacting with a symmetrical scalar or pseudoscalar field is investigated in the case of non-relativistic motion of the nucleons. An upper bound is obtained for the energy of the ground state of the system. This is found to tend to $-\infty$ in the case of a pseudoscalar field with pseudovector coupling, so that no ground state with finite energy exists in this case. *N. Rosen* (Haifa).

Leigh, R. S. The augmented plane wave and related methods for crystal eigenvalue problems. Proc. Phys. Soc. Sect. A. **69** (1956), 388–400.

In the augmented plane wave method, and in an earlier related method, both due to Slater, the wave functions in a crystal are expressed in terms of spherical harmonics and radial functions around each atom, and of plane waves between the atoms. Convergence in these two methods with respect to the addition of l values near the atoms is discussed in the light of a variation principle which involves an arbitrary parameter. It is shown that this convergence is likely to be improved by calculating the matrix elements in a way slightly different from that given by Slater. This is illustrated by a simple example. A new method, of the same general kind, is suggested, which has some advantages over the other two methods.

Author's summary.

Nishiyama, Toshiyuki. Theory of sound waves and collective description. Progr. Theoret. Phys. **14** (1955), 37–51.

A system consisting of a large number of electrons is treated by means of the linearized equation for the density matrix, the solution to which is given by a set of normal coordinates. By means of a canonical transformation the collective motion is separated out from the remaining motion of the system. The interaction between the electrons and a lattice is investigated. The conditions for the validity of the approximations used, such as the assumption that excited particles and holes are restricted to the neighborhood of the Fermi surface, are discussed. A number of results obtained are compared with those of other authors. *N. Rosen* (Haifa).

Yoccoz, J. Modèle en couches et mouvements collectifs. J. Phys. Radium (8) **17** (1956), 517–518.

Une méthode variationelle, proposée par Hill et Wheeler, est utilisée pour décrire les mouvements collectifs du noyau. Les résultats, en ce qui concerne la relation entre le moment d'inertie I du noyau et sa déformation, sont meilleurs que ceux obtenus avec le modèle hydrodynamique. *Résumé de l'auteur.*

Lee, T. D.; and Yang, C. N. Charge conjugation, a new quantum number G, and selection rules concerning a nucleon-antinucleon system. Nuovo Cimento (10) 3 (1956), 749–753.

It is well-known that selection rules can be derived from the conservation of charge parity for neutral systems, if the interactions involved are invariant under charge conjugation. A similar type of parity associated with extended charge conjugation — the product of charge conjugation and charge symmetry — may be defined for systems of zero nucleon number, if the interactions are charge independent. The consequent selection rules for the decay of the nucleon-antinucleon system are tabulated.

If 'strangeness', but not total isotopic spin, is conserved, similar considerations apply to states of zero 'strangeness' and zero baryon number. *P. T. Matthews.*

Wilson, Richard. Polarization in nucleon scattering at various energies. Phil. Mat. (7) **46** (1955), 769–782.

The scattering of high energy (~ 100 Mev) nucleons by nuclei is generally studied by means of a model which assumes that the interaction of the incoming nucleon with the nucleus is well represented by a complex field of force. This complex potential has a spin orbit coupling term which has to explain the experimentally observed polarization effect [Fermi, Nuovo Cimento (9) **11** (1954), 407–411].

It is, however, too difficult to solve exactly the Schrödinger equation of the nucleon in the external potential. An "optical" approximation, which uses the fact that the neutron wave length is small as compared to the nuclear radius, seems very reliable for such a problem [Fernbach, Serber, and Taylor, Phys. Rev. (2) **75** (1949), 1352–1355; Malenka, ibid. **95** (1954), 522–526].

In the paper reviewed here the optical model is used as a basis for the study of the variation of the polarization as a function of energy. It is shown that experimental evidence is in favour of a spin orbit term of the form of the Thomas correction (i.e. the potential is large only on the nuclear surface). The validity of the approximations used is discussed carefully. *S. Fubini* (Chicago, Ill.).

Sen, P. Renormalized Dirac Maxwell equations. Nuovo Cimento (10) 4 (1956), 270–282.

The author continues his previous program [Nuovo Cimento (10) 3 (1956), 390–408, 612–625; MR 17, 1164] and studies the details of the renormalization of the coupled Dirac Maxwell equations. It appears that the author's method allows several alternative formulations of the renormalization program, none of which seems to agree completely with conventional theory in higher orders. As the second order expressions do agree with what is usually obtained, the author considers his scheme not to be in disagreement with experimental facts. No discussion of general problems like the unitarity of the S-matrix or of causality is given, neither in this nor in the previous papers. As pointed out in the review of the first paper cited above these questions give rise to serious trouble in the kind of theory considered by the author.

G. Källén (Copenhagen).

Potier, Robert. Sur la théorie des champs quantifiés en relativité généralisée. C. R. Acad. Sci. Paris 243 (1956), 939–942.

The author proposes a theory of a quantized spinor field, ψ, and an unquantized gravitational field, $g_{\mu\nu}$, satisfying the equations

$$\gamma^{\mu}\psi_{\mu}=\kappa\psi, \tag{1}$$

$$i\,\frac{\delta\Psi}{\delta\sigma(x)}=\mathfrak{H}(x)\Psi, \tag{2}$$

$$R_{\mu\nu}(x)-\tfrac{1}{2}g_{\mu\nu}(x)R(x)=\kappa(\Psi,\,T_{\mu\nu}(x)\Psi). \tag{3}$$

(1) is the Dirac equation in general relativity theory. (2) is a Tomonaga-Schwinger equation which governs the time dependence of the state vector Ψ. The interaction hamiltonian density, $\mathfrak{H}(x)$, is constructed in accordance with a previous note [same C. R. 242 (1956), 1694–1697, MR 17, 929]. (3) is Einstein's equation for the case in which the source of the gravitational field is the expectation value of the stress-energy-momentum density of the spinor field. The author gives three necessary conditions for the compatibility of (1), (2), and (3), {but does not discuss the existence of solutions or the difficulties of principle arising from the partial invalidity of the uncertainty principle.}

A. S. Wightman.

Kalitzin, Nikola St. Untersuchungen über dem magnetischen Moment des Nukleons (relativistische Wellengleichung des Nukleons). Trudy Vysšego Inst. Narod. Hozyaĭstva Staline. Inž.-Stroitel. Fak. 1 (1954), 1–64. (Bulgarian. German summary)

The Dirac equation is generalized to include isobaric spin. The resultant linear equation for the eight-component nucleon wave function is based on a six-dimensional space and involves eight 8×8 "Dirac"-matrices. (A special representation is used throughout.) This equation is discussed and it is pointed out that it leads to a mass spectrum. The various bilinear covariants are constructed. Among these are antisymmetric tensors of second rank which can be interpreted as magnetic moment densities associated with ordinary spin and isobaric spin respectively. Using an obscure definition of the nuclear magnetic moment the author finds that for the proton the spin magnetic moment is $+\sqrt{2}$ and the isobaric spin magnetic moment is $+1$ (in units of nuclear magnetons). For the neutron the opposite signs are obtained. Comparing half the difference of the observed proton and neutron moments, 2.35, with $1+\sqrt{2}$ the author claims: "Eine bessere Übereinstimmung mit dem Experiment ist aus einer symmetrischen Theorie kaum zu erwarten".

F. Rohrlich (Iowa City, Ia.).

Pappalardo, R. Su una nuova equazione relativistica dell'elettrone proposta da Zaitsev. Nuovo Cimento (10) 4 (1956), 166–167.

G. A. Zaitsev [Ž. Eksper. Teoret. Fiz. 28 (1955), 530–540; MR 17, 330] has proposed a "new" relativistic equation for the electron. It is shown that the Zaitsev equation coincides completely with the well-known Dirac equation.

S. Fubini (Chicago. Ill.).

Castoldi, Luigi. Le equazioni di Dirac dedotte dalla legge di conservazione della particella, e loro carattere relativistico generale. Rend. Sem. Fac. Sci. Univ. Cagliari 26 (1956), 90–95.

Castoldi, Luigi. Il teorema di Ehrenfest nella meccanica quantica relativistica della particella. Rend. Sem. Fac. Sci. Univ. Cagliari 26 (1956), 96–104.

See also: Bogolyubov and Parasyuk, p. 404; Rubin and Shuler, p. 429.

Relativity

Szamosi, G. Variational principle and potential in relativistic dynamics. Acta Phys. Acad. Sci. Hungar. 6 (1956), 207–215. (Russian summary)

The purpose of this paper is to develop a Lorentz-invariant canonical method of relativistic dynamics. The usual difficulty lies in the fact that the four velocity vector components u_r are not independent from each other but satisfy the condition $u_r u_r=1$. The author's trick consists in the following: he uses as a parameter not τ ($d\tau^2=dx_r dx_r$), but an arbitrary invariant s; then the formalism becomes as simple as that in classical mechanics. After having the equations of motion τ is reintroduced through the equation $ds/d\tau=m_0/M$, where m_0 is constant (the mass in absence of external field) and M is the "rest mass", generally not constant.

L. Infeld.

Frisch, David H.; and Wilets, Lawrence. Development of the Maxwell-Lorentz equations from special relativity and Gauss's law. Amer. J. Phys. 24 (1956), 574–579.

Maxwell's equations and the Lorentz force law are developed by applying the transformations of special relativity to Gauss's law on the flux of the electric field, $\int E\cdot dS=4\pi q_{\text{enclosed}}$, provided that electromagnetic signals are propagated with the speed of light, and that the electric force on a test charge is velocity-independent and does not depend on time derivatives of the source position of order higher than the acceleration.

Author's summary.

★ **Фок В.А. [Fok, V.A.]. еория пространства времени и тяготения. [Theory of space, time and gravitation.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1955. 504 pp. 16.85 rubles.

This book has two quite separate aspects. First, it is a text-book for the use of graduate students in theoretical physics, giving a complete exposition of the theories of special and general relativity. Second, it is an eloquent and at times somewhat polemical plea for an unorthodox

interpretation of Einstein's theory of gravitation. It is convenient to discuss these two aspects of the book separately. First, the text-book aspect.

There are seven chapters, covering the following ground. (1) Logical development of the special theory of relativity, starting from the principle of Galilean invariance and the equations of Maxwell. This includes two original features, a rigorous proof that the allowed coordinate transformations in special relativity must be linear, and a full discussion of the phenomenon of stellar aberration using a representation in which velocities are points in a Lobačevskiĭ hyperbolic 3-space. (2) Tensor analysis in special relativity. Tensors are defined, and the mechanics of systems of charges interacting with the Maxwell field is developed in detail. (3) Tensor analysis in generalized coordinates. Definitions of covariant differentiation, the curvature tensor, etc. This is the only chapter in which the author has nothing new to say. (4) The mechanics of masses and charges in special relativity is reexamined using generalized coordinates. (5) Foundations of the theory of gravitation. The Einstein field equations are induced by formal generalization of Newton's law of gravitation, starting from the hypothesis that the geometry of space-time is Riemannian. The field equations are immediately simplified by making a special choice of coordinates, first introduced by T. De Donder [Ann. Observ. Roy. Belg. (3) 1 (1921), 73–268]. The special coordinates are called "harmonic" because they satisfy the equations

$$\Box x_\alpha = 0, \tag{1}$$

where $\Box$ denotes the covariant D'Alembert operator. An equivalent statement of the harmonic condition is

$$(\partial/\partial x_\beta)[(-g)^{\frac{1}{2}} g^{\alpha\beta}] = 0. \tag{2}$$

Various special problems are solved using harmonic coordinates. It is proved that gravitational signals are propagated with velocity c. The clock paradox is discussed. (6) Equations of motion are derived from a variational principle, first for point particles and then for extended material bodies. Matter is always represented by an energy-momentum tensor and not by singularities of the metric. The equations of motion for a system of rigidly-rotating bodies in gravitational interaction with one another are calculated explicitly in a power-series development, considering the velocities to be small compared with c. With a comparatively modest amount of work the results are computed as far as terms in $(v/c)^2$ and including three-body interactions of the form $[\gamma^2 m_1 m_2 m_3 / r^2 c^2]$. The double-star problem is solved as a special case of this analysis. The results of this chapter relating to the effects of internal structure and rotation on the equations of motion are new. (7) The gravitational potentials existing at a large distance from a system of moving masses are calculated. As in electrodynamics, the results take a different form in the "near zone" and in the "wave zone". Of special interest are the wave-zone potentials which give directly the energy emitted by the system of masses in the form of gravitational waves. In particular, the emission from the solar system (mainly due to the motion of Jupiter) is calculated and found to have the value 450 watts, or 10^{-24} of the Sun's electromagnetic radiation. The use of harmonic coordinates plays an essential part in these calculations in two ways. First, the algebraic complexity of the equations is greatly reduced. Second, the difficulties of principle, which arise when one tries to define gravitational radiation in terms of general coordinates, are completely resolved. A satisfactory definition of gravitational radiation becomes possible because of the following uniqueness theorem which is proved with some degree of rigor. Let x_α and x'_α be two harmonic coordinate systems defined in the same physical space-time. Suppose

$$x'_\alpha = a_\alpha + a_{\alpha\beta} x_\beta + \eta_\alpha(x_1, x_2, x_3, x_4), \tag{3}$$

where the η_α with their first derivatives are everywhere bounded and decrease like $(1/r)$ at infinity. Let $g_{\alpha\beta}$, $g_{\alpha\beta}'$ be the metric tensors in the two systems, and suppose that the difference $(g_{\alpha\beta} - g_{\alpha\beta}')$ represents asymptotically only outgoing waves at infinity. Then $\eta_\alpha = 0$, so that the two coordinate systems are related by a linear Lorentz transformation. The proof of this theorem is given in detail only for the case of a flat space-time, with a remark that the general case can probably be handled similarly. In conclusion there is a short discussion of the cosmological problem and of the historical development of general relativity.

Now we turn to the controversial aspect of the book, namely the author's peculiar attitude to the physical interpretation of general relativity. This question is discussed in an 8-page introduction, in a 2-page concluding summary, and at intervals throughout the text, particularly in Sections 61 and 96. He violently objects to the "principle of equivalence" which Einstein regarded as the corner-stone of the theory. He objects to the notion that invariance under general coordinate transformations is a property of the physical universe. He regards this invariance as merely a property of the mathematical symbolism, which it may be sometimes convenient to maintain and sometimes inconvenient. He objects to the use of the name "general relativity" for Einstein's theory of gravitation. He defines "relativity" to mean an invariance in the description of nature as viewed from different physical observation-points (not coordinate-systems). In this sense the gravitation theory contains less relativity than what is usually called "special relativity", and not more. So he vehemently insists that we call "special relativity" "relativity", and "general relativity" "gravitation theory."

There are two main arguments against the physical importance of general coordinate invariance. (1) He shows by examples that it is easy to rewrite various physical theories, without in any way changing their physical content, in a generally covariant form. For example, the Lagrangian equations of motion for a system of non-relativistic particles can be so rewritten. Another example is the translation of "special relativity" into general coordinate notations in Chapter 4. The point in these examples is that a "metric tensor" is introduced into the formalism in such a way as to nullify the effects of coordinate transformations on all physically observable quantities. The invariance of the theory is thus purely formal and has no physical significance. (2) For the specification of any concrete situation in gravitation theory, one needs not only the Einstein field equations but also the boundary conditions to be satisfied by the fields at large distances. Though the field equations have general coordinate invariance, the boundary conditions definitely have not. An important example of this is the ambiguity mentioned earlier in the definition of gravitational radiation in terms of general coordinates. The boundary conditions which define the presence or absence of incoming radiation can only be stated in a special class of coordinate systems. The general invariance

of the field equations is thus a local property which disappears as soon as one tries to specify a physical situation in the large. The author believes, on the basis of his unrivalled experience with particular solutions of the Einstein equations, but without any general mathematical proof, that the only coordinate systems in which the fields can be unambiguously specified in the large are harmonic systems. The theorem stated above establishes that any two harmonic coordinate systems are Lorentz equivalent. Thus the formulation of the gravitation theory in terms of harmonic coordinates re-establishes the Lorentz group as the true measure of the "relativity" permitted in a satisfactory description of the physical world.

The principle of equivalence states that every gravitational field can be reduced locally to zero by an appropriate choice of accelerated coordinate system. The author maintains that this principle is just as empty of meaning as the principle of general coordinate invariance of which it is a special case. Therefore Einstein's use of the principle of equivalence as a reason for postulating the more general principle of coordinate invariance does not add weight to either principle. In concrete terms, the principle of equivalence always fails when it is extended to nonlocal phenomena (e.g. the gravitational field of the earth is not equivalent to any acceleration field in the large, and the acceleration field produced by the earth's rotation is not equivalent to any gravitational field). When restricted to purely local events, the principle of equivalence does not state more than the equality of gravitational and inertial mass.

In the author's view, the basic hypothesis of the gravitation theory is that the geometry of space-time is Riemannian. This hypothesis he regards as an extrapolation from two physical laws, (1) the law of propagation of light signals in special relativity, and (2) the equality of inertial and gravitational mass. After the Riemannian geometry is postulated, he then regards it as a mathematical accident that the local behavior of the metric is conveniently described by means of general coordinate systems.

{The reviewer's opinion is that the author has a good case but has overstated it. It may well be true that general coordinate invariance has meaning only locally. But the most important consequence of general invariance is the existence of local conservation laws for energy and momentum, and these are valid whether general invariance holds in the large or not. The reviewer feels that the author has made a major contribution to the understanding of gravitation theory, especially by his insistence on studying the solutions of the field equations and not merely the formal properties of the equations. The future will probably confirm the validity both of the author's point of view and of Einstein's, each in its proper sphere.}

F. J. Dyson (Princeton, N.J.).

Słowikowski, W. Note on the application of the Pauli ring to form the metric tensor in the general theory of relativity. Bull. Acad. Polon. Sci. Cl. III. **4** (1956), 313–320.

Spin-tensors (called in the paper P-tensors) are considered as elements of the Pauli ring. An anholonomic coordinate system is used to introduce a metric tensor, a non-symmetric connexion and an "invariant" derivative. {This derivative is rather meaningless, being non-covariant. The author's formalism differs only in notation from that used in the spinor calculus.} *A. Trautman.*

Maurer-Tison, Françoise. Sur les coordonnées isothermes en théorie unitaire. C. R. Acad. Sci. Paris **243** (1956), 1196–1198.

The main achievement is the formula of approximation for the contracted curvature tensor $R_{\mu\lambda}$ of the Einstein unified field theory. The approximation is of the third order, the final formula express $R_{\mu\lambda}$ in terms of the basic tensor $g_{\mu\lambda}(\neq g_{\lambda\mu})$ and its derivatives. The coordinates are not general. {Remark of the reviewer. A fast converging approximation of much higher order and in general coordinates can easily be obtained by the method displayed in Hlavatý, J. Rational Mech. Anal. 2 (1953), 1–52 [MR 14, 505].} *V. Hlavatý* (Bloomington, Ind.).

See also: Stanyukovič, p. 440; Potier, p. 445; Castoldi, p. 445.

Astronomy

Sconzo, Pascual. Perturbations in heliocentric coordinates calculated by a numerical procedure of analytic prolongation. Rev. Un. Mat. Argentina **17** (1955), 223–229 (1956). (Spanish)

The computation of special perturbations by a third body in the solar system is usually carried out by evaluating the perturbing forces in rectangular coordinates point by point and then performing a numerical integration of the accelerations. In 1942 Stumpff evaluated Jupiter perturbations on asteroids by successively expanding the position of the perturbed body in Taylor series of time and separating the principal terms from the remainder, which he proceeded to integrate numerically.

This paper is a procedural variation on Stumpff's method. The author takes the analytical expressions of the Taylor expansions in three coordinates up to the fourth-order terms and, making use of the Lagrangian functions f and g, obtains expressions for x, y and z which can be computed directly, without numerical integration. This method will give perturbed positions at intervals such that fifth-order terms may be safely neglected. One of its merits lies in the fact that no use is made of a fictitious "unperturbed" orbit on which perturbations are superimposed: the perturbed body is steered on its perturbed orbit right from the beginning. *L. Jacchia.*

Gazarhi, L. A. On a case of plane motion of three material points. Ukrain. Mat. Ž. **8** (1956), 208–213. (Russian)

The author considers one special case of the plane motion of three material points (P_i) under the action of the forces $|F_{ij}|=m_im_j|f(r_{ij})|$, $i, j=0, 1, 2, i\neq j$, where $f(r_{ij}=\overline{P_iP_j})$ is the function of the mutual distance of two points, with the condition that the ratio $r/\rho=$const; r is the distance between two points and ρ the distance of the 3rd point from the centre of inertia of two preceding points. Using the moment of inertia of a system as independent variable, and profiting by the integrals of energy ($h=$const) and of angular momentum ($c=$const), the problem is reduced to the integration of a system of four differential equations of the 1st order and to two quadratures. It is shown that if there exists a class of functions of the form $f(r)=\sum_{-\infty}^{\infty} C_\nu r^{\nu\alpha}$, then the solution of the differential equations has the form $f(r)=Ar+Br^3$. In the case $B\neq 0$ the motion is possible if $c=0$ and $m_i=m$; then $r/\rho=2/\sqrt{3}$. *D. Rašković* (Belgrade).

Vandakurov, Yu. V. On a method of approximate solution of the problem of n bodies in natural coordinates. Byull. Inst. Teoret. Astr. 6 (1955), 240–243. (Russian)

The author proposes a method for the solution of the n-body problem in the first approximation. This method is based on the determination of trajectories of the perturbed motion in terms of coordinates connected with the trajectory of the unperturbed motion. The final result is obtained by quadratures. *E. Leimanis.*

Merman, G. A. On a theorem of Birkhoff. Byull. Inst. Teoret. Astr. 6 (1955), 232–239. (Russian)

The author reformulates more precisely and proves two theorems of G. D. Birkhoff [Dynamical systems, Amer. Math. Soc. Colloq. Publ., vol. 9, New York, 1927, p. 282], namely, that in the problem of three bodies for $h<0$ (h being the energy constant) the motions are (i) of hyperbolic-elliptic type and (ii) of hyperbolic-elliptic type and of the same class, both for $t\to+\infty$ as well as for $t\to-\infty$. He also gives criteria which guarantee the realization of motions of the considered types and proves a corollary of another theorem of Birkhoff (last theorem on p. 276 of the same book) to the effect that the ratio of the largest to the smallest of the mutual distances can be made arbitrarily large throughout the motion by giving a sufficiently large value to Birkhoff's constant K [cf. also Merman, same Byull. 5 (1954), 594–605; MR 17, 905].

E. Leimanis (Vancouver, B.C.).

Message, P. J. The second-order theory of the figure of Jupiter. Monthly Not. Roy. Astr. Soc. 115 (1955), 550–557 (1956).

Die hydrostatische Theorie zweiter Ordnung für rotierende Planeten wird auf die vier Massenverteilungen angewendet, die Ramsey und Miles für Jupiter angegeben haben. Für jeden der Fälle werden die Koeffizienten im Außenfeld gefunden, ferner wird eine (bisher für streng ellipsoidische Planeten bekannte) Ungleichung für diese Koeffizienten erweitert auf den Fall, daß Abweichungen bis zur zweiten Ordnung vom Ellipsoid vorliegen. Vergleich mit Beobachtungen. *K. Maruhn* (Dresden).

Elsasser, Walter M. Hydromagnetic dynamo theory. Rev. Mod. Phys. 28 (1956), 135–163.

The author discusses from the viewpoint of the hydromagnetic equations the generation and maintenance of cosmic magnetic fields. *C. H. Papas* (Pasadena, Calif.).

See also: Stanyukovič, p. 440; McMinn, p. 449.

Geophysics

Dmitriev, A. A. On the modelling of geophysical circulation for a rotating parabolic vessel. Izv. Akad. Nauk SSSR. Ser. Geofiz. 1956, 320–326. (Russian)

L'auteur déduit les équations du mouvement d'un liquide en rotation par rapport à un système des coordonnées exponentielles et paraboliques. L'auteur montre que pour des distances peu éloignées de l'axe de rotation les équations du mouvement dans le système des coordonnées exponentielles-paraboliques différent peu des équations du mouvement rapportées aux coordonnées sphériques. *M. Kiveliovitch* (Paris).

Plotkin, B. I. Certain questions of the theory of groups without torsion. Ukrain. Mat. Ž. 8 (1956), 325–329. (Russian)

The intersection of all maximal isolated subgroups of a torsion-free group is a subgroup with properties analogous to those possessed by the ϕ-subgroup in a finite group. Using the study of such properties as a foundation, the author characterizes, in terms of the lattice of all subgroups of a group, the condition that the group be a nilpotent torsion-free group of rank n. Finally the free nilpotent group with finitely many generators is shown to be determined up to within isomorphism by the lattice of its subgroups. *R. A. Good* (College Park, Md.).

Racer-Ivanova, F. S. Investigation of free oscillations of a fluid of diurnal or semi-diurnal type in shallow-water basins. I. Izv. Akad. Nauk SSSR. Ser. Geofiz. 1956, 67–78. (Russian)

Es werden die freien Schwingungen in genügend seichten Meeren auf der Erdoberfläche untersucht, wobei die Schwingungen vom Tages- und Halbtagestyp im Vordergrund stehen und die Reibung vernachlässigt wird. Die horizontale Geschwindigkeitskomponente wird unabhängig von der Meerestiefe angenommen, die vertikale Komponente soll klein sein gegenüber der horizontalen. Die Bewegungsgleichung für die Verschiebung läßt sich wegen ihrer Verwandtschaft mit der Besselschen Differentialgleichung nach Umformung in eine Integralgleichung numerisch behandeln. Beispiel. *K. Maruhn.*

Andreoletti, J. Les modèles en météorologie dynamique. J. Sci. Météorol. 8 (1956), 29–46. (Spanish summary)

The author takes the equations of motion of a plane atmosphere, including the effect of viscosity, and rewrites them in a form more suitable for numerical forecasting. Models of one, two, and three parameters are discussed but no calculations and comparison with observational data are attempted. *M. H. Rogers.*

Tatarskiĭ, V. I. Microstructure of the temperature field near the ground. Izv. Akad. Nauk SSSR. Ser. Geofiz. 1956, 689–699 (1 plate). (Russian)

L'étude théorique de la microstructure du champ de températures dans un courant turbulent a été étudie par A. M. Obuhov [mêmes Izv. 1951, no. 3, 49–68] et A. M. Yaglom [Trudy Geofiz. Inst. no. 24(151) (1954), 112–162].

L'auteur expose certains résultats théoriques de M. Obuhov. Il expose ensuite les nouvelles méthodes de mesures de la fonction du champ des températures et ses résultats. *M. Kiveliovitch* (Paris).

Sneyers, R. Sur la détermination de l'homogénéité des séries climatologiques. J. Sci. Météorol. 7 (1955), 359–372. (Spanish summary)

L'auteur reprend l'idée de V. Conrad sur la détermination de l'homogénéité des séries climatologiques et passe en revue différents tests: variable aléatoire gaussienne et le cas non paramétrique.

Quelques exemples sont donnés: 1) mesure de la vitesse du vent à Uccle, 2) la côte udometrique du Brabant, 3) la mesure de la température de l'air à Uccle.

M. Kiveliovitch (Paris).

Tatarskiĭ, V. I. On the amplitude and phase pulsations of a wave moving in a slightly unhomogeneous atmosphere. Dokl. Akad. Nauk SSSR (N.S.) 107 (1956), 245–248. (Russian)

Ce problème a été déjà étudié par A. M. Obuhov [Izv.

Akad. Nauk SSSR. Ser. Geofiz. **1953**, 155–165; MR, **15** 1002; **17**, 798]. La solution obtenuep ar M. Obuhov est malheureusement très compliquée et difficilement applicable pratiquement. L'auteur propose une solution plus simple qui se prête mieux pour les applications pratiques.
M. Kiveliovitch (Paris).

Kuo, H.-L. On quasi-nondivergent prognostic equations and their integration. Tellus **8** (1956), 373–383.

In the first part of this paper, the author derives an approximation to the wind system which is better than the geostrophic approximation. This is done by requiring the horizontal divergence to remain small compared with the vertical component of the relative vorticity for large scale motions. This can be expressed by a balance equation which, when the nonlinear terms in it are neglected, reduces to the geostrophic relation. This balance equation is used in deriving forecasting equations, and the second part of the paper is devoted to some general methods of integrating these equations. *M. H. Rogers.*

Oblogina, T. I. Dynamic characteristics of diffracted elastic waves. Izv. Akad. Nauk SSSR. Ser. Geofiz. **1956**, 377–390. (Russian)

The author determines the dynamical hodograph of elastic waves diffracted by various geometrical configurations. Both two- and three-dimensional cases are considered, and the results are depicted in several graphs. Theoretical seismograms of different pulse forms (impinging) are also obtained. The mathematical method of analysis is attributed to A. F. Filippov [Dissertation, Moskov. Gos. Univ., 1953; not available to the reviewer]. Some aspects of the present work are also related to Skuridin's methods [Izv. Akad. Nauk SSSR. Ser. Geofiz. **1955**, 3–16; MR **17**, 319]. *K. Bhagwandin* (Oslo).

McMinn, Trevor J. On figures of equilibrium of rotating liquids. Math. Z. **64** (1956), 286–297.

Gegeben sei ein Flüssigkeitskörper konstanter Dichte, der um eine raumfeste Achse rotiert und nur den eigenen Newtonschen Anziehungskräften unterworfen ist. Der Körper ist bezüglich jeder Parallelen zur Rotationsachse konvex und besitzt eine zu dieser Achse senkrechte Symmetrieebene. Dies ist insbesondere für Gleichgewichtsfiguren rotierender Flüssigkeiten erfüllt. Durch Modifizierung eines Verfahrens von Lichtenstein gelingt es, eine Schranke für den Krümmungsradius in Äquatorpunkten von Kurven zu finden, die durch zur Rotationsachse parallele Ebenen aus Flächen konstanten Gesamtpotentials herausgeschnitten werden. Die Ergebnisse gestatten Schranken für die Abplattung einer Gleichgewichtsfigur zu finden, die gegenüber den bisher bekannten ($\sqrt{20}$ bei Nikliborc) kleiner sind. Für recht allgemeine Klassen von Figuren wird die Schranke 2, bzw. bei weiterer Einschränkung $\sqrt{3}$ erhalten. Besitzt die Figur eine die Rotationsachse enthaltende Symmetrieebene und ist sie konvex in der zu dieser senkrechten Richtung, so ist die Abplattung bezüglich dieser Richtung kleiner oder gleich eins. *K. Maruhn* (Dresden).

Renner, J. Untersuchungen über Lotabweichungen. Acta Tech. Acad. Sci. Hungar. **15** (1956), 37–75. (Russian, English and French summaries)

The study of deviations of the vertical are important for a geodesist since they allow a better determination of the geoid's surface. Eötvös showed how to compute them from data of a torsion balance survey [Verh. 15. Allg. Konferenz Internat. Erdmessung, Budapest, 1906, Berlin-Leyden, 1908, pp. 337–397]. His method is much simplified and improved in this paper. The procedure used necessitates the interpolation of torsion balance data at the grid points of a square network.

An interesting numerical example suggests that the transformation of torsion balance data into deviations of the vertical could be useful also in the interpretation of gravimetric anomalies especially of those related to orebodies. *E. Kogbetliantz* (New York, N.Y.).

Holsen, Jon. Das mittlere Fehlerellipsoid. Schweiz. Z. Vermessg. Kulturtech. Photogr. **54** (1956), 266–278.

The author generalizes to three variables (x, y, z) the classic two-dimensional problem in geodetic triangulation adjustments of determining the extremal values M_i of the mean error $M(\phi)$ of an adjusted position P as a function of the direction ϕ emanating from P. M_i can be geometrically represented as the axes of an ellipse $(i=1, 2)$ in the two-dimensional case. [For a solution of the problem in two variables cf. e.g. W. Grossmann, Grundzüge der Ausgleichungsrechnung$\cdots$, Springer, Berlin, 1953; MR **15**, 650.]

It is assumed that the six weight coefficients Q_{jk} $(j, k=x, y, z)$ are given from the adjustment. Let α_i $(i=1, 2, 3)$ be the direction angles of a direction from P. It is first proved that M_i $(i=1, 2, 3)$ can be geometrically represented as the axes of an ellipsoid. Then an expression for $M(\alpha_i)$ is set up as a function of Q_{jk} and α_i. Using the relation $\cos^2 \alpha_1+\cos^2 \alpha_2+\cos^2 \alpha_3=1$ as a condition equation, a Lagrangian multiplier is employed to obtain M_i and the corresponding values of α_i. It is mentioned that M_i can be derived also by a transformation of axes without applying the calculus (and indeed is done in this way by Grossmann in the two-dimensional case). The author also discusses special cases arising from restrictions placed on Q_{jk}; e.g., $Q_{yz}=Q_{xz}=0$ implies that there is no correlation between the horizontal and vertical angle measurements in geodetic work. The general results are illustrated by a numerical example.
B. Chovitz (Washington, D.C.).

See also: Mauersberger, p. 388; Thompson, p. 416; Krarup and Svejgaard, p. 421; Kawasumi, p. 421; Pincus, p. 427; Slobodov, p. 439.

OTHER APPLICATIONS

Games, Economics

★ **Wolfe, Philip. Determinateness of polyhedral games.** Linear inequalities and related systems, pp. 195–198. Annals of Mathematics Studies, no. 38. Princeton University Press, Princeton, N. J., 1956. $5.00.

The author considers the question whether

$$\sup_x \inf_y yAx=\inf_x \sup_y yAx,$$

where A is an m by n matrix and x and y are constrained within arbitrary closed convex polyhedra; the values $\pm\infty$ are admitted. If the equation is valid and the value is finite, then optimal strategies x and y exist; and this case is equivalent to the feasibility of both of two associated linear programming problems. The equation can fail only if sup inf is $-\infty$ and the inf sup$+\infty$.
J. Isbell.

★ **Mills, Harlan D. Marginal values of matrix games and linear programs.** Linear inequalities and related systems, pp. 183–193. Annals of Mathematics Studies, no. 38. Princeton University Press, Princeton, N. J., 1956. $5.00.

The derivative of the values of (i) a matrix game, (ii) a linear programming problem are shown to be the values of certain associated (i) games, (ii) linear programming problems. Precisely, if A is a matrix game, and H a matrix of the same size as A, then

$$\lim_{\alpha\to 0+} (\text{value } [A+\alpha H]-\text{value } [A])/\alpha$$

always exists and is the value of the constrained matrix game whose payoff matrix is H and whose spaces of strategies are the sets of optimal strategies of the unconstrained game with payoff A. For (ii), observe that existence of a value is not guaranteed. Suppose $\Phi(A)$ is the maximum value of $a_{00}+\sum x_i a_{i0}$ subject to $x_i \geqq 0$ and $a_{0j}+\sum x_i a_{ij} \geqq 0$ (i, j not zero). Assume that for some $\alpha_0>0$, for $0\leqq\alpha<\alpha_0$, $\Phi(A+\alpha H)$ exists. Then the derivative of Φ at A in the direction $+H$ exists and is given by a certain minimax expression; also, the author constructs a matrix B such that if (also) $\Phi(B)$ exists then $\Phi(B)$ is the indicated derivative of $\Phi(A)$. Similar results are indicated for linear programming problems presented in different forms.

The case in which H has only one non-zero entry gives especially simple formulas (discovered by O. Gross in the case of games, as the author notes). For linear programming problems a connection is noted between these formulas and known properties of Lagrange multipliers. Some unsolved problems are stated concerning the existence of $\Phi(A)$.

{Reviewer's remarks: The second unsolved problem appears to be a slip, Theorem 2 answering the question. Since almost all games have unique solutions, Theorem 1 shows that the value function is smooth a.e. in the sense of having a gradient which determines the derivative in all directions.} *J. Isbell* (Princeton, N.J.).

Blackwell, David. An analog of the minimax theorem for vector payoffs. Pacific J. Math. 6 (1956), 1–8.

Let X be a compact set in Euclidean N-space, $P\,[Q]$ be the set of vectors $(p_1, \cdots, p_r)$ $[(q_1, \cdots, q_s)]$ of non-negative components whose sum is 1, and $M=\|m(i,j)\|$ an $r\times s$ matrix each element of which is a probability distribution on X. By a strategy of player I [II] is meant a sequence $f=(f_n)_0^\infty$ $[g=(g_n)_0^\infty]$ of functions f_n $[g_n]$ with domains $(x_1, \cdots, x_n)$, $x_i \in X$, and range in $P\,[Q]$ (f_0 $[g_0]$ being constant). M and a pair (f, g) determine a sequence of random variables $x_1, \cdots, x_n, \cdots$ as follows: i $[j]$ are chosen according to f_0 $[g_0]$ and then x_1 is selected according to $m(i, j)$; and having determined x_n, i $[j]$ are chosen according to $f_n(x_1, \cdots, x_n)$ $[g_n(x_1, \cdots, x_n)]$ and then x_{n+1} according to $m(i, j)$.

Let S be a set in N-space and denote by δ_n the distance of $(x_1+\cdots+x_n)/n$ from S. S is said to be approachable [excludable] in M if there exists f^* $[g^*]$ such that $\lim \delta_n=0$ [$\liminf \delta_n>0$] with probability 1, uniformly for all (f^*, g) $[(f, g^*)]$. A necessary condition for approachability is established which enables to deduce that every convex S is either approachable or excludable. For $N=1$ all sets are either approachable or excludable and a simple characterization of both kinds is given. An example shows that for $N=2$ there exist bounded sets which are neither approachable nor excludable, and it is conjectured that for every N each set is either weakly approachable or weakly excludable, where the weak concepts are obtained by replacing the probability 1 requirement above by that of stochastic convergence of δ_n to 0 [boundedness away from 0]. *A. Dvoretzky* (Jerusalem).

Uzawa, Hirofumi. A generalization of Laplace criterion for decision problems. Ann. Inst. Statist. Math., Tokyo 7 (1956), 123–129.

The author deals with a generalization of a result of Chernoff [Econometrica 22 (1954), 422–443; MR 16, 271]. In the generalized problem the set of states of nature need not be finite but may be a topological space Ω on which a group Σ of transformations is defined. Also a measure μ exists on Ω which is invariant with respect to Σ. Then under certain postulates of rational selection, the author finds that such a selection yields a strategy x_0 which maximizes $\int_\Omega x(t)\mu(dt)$, where $x(t)$ is the utility corresponding to strategy x and state t. This result is the natural extension of the Laplace criterion obtained by Chernoff for n states of nature and by Uzawa for the case where Ω was n dimension Euclidean space in a paper submitted to Econometrica. The postulates used represent a form of "total ignorance" about the state of nature. *H. Chernoff* (Stanford, Calif.).

Burger, E. Zur Theorie der kooperativen Zweipersonenspiele. Arch. Math. 7 (1956), 143–147.

Nash [Econometrica 21 (1953), 128–140; MR 14, 778] and Raiffa [Contributions to the theory of games, v. 2, Princeton, 1953, pp. 361–387; MR 14, 667] have independently proposed a theory of cooperative two-person games. The players select strategies independently which each will use if negotiated cooperation with the other player is not possible. Then they bargain with each other to obtain a mutually more profitable pay-off. The author provides another proof that the negotiated pay-off can be determined uniquely by the independently selected "threat" strategies through a not necessarily unique equilibrium point. A feature of the proof is that it uses the Brouwer fixed-point theorem only. There are misprints in the form of some misplaced tildes in lines 26, 27, and 28 of page 145, but it is clear from context what the corrections should be. *E. D. Nering* (Tucson, Ariz.).

Dantzig, George B. Constructive proof of the Min-Max theorem. Pacific J. Math. 6 (1956), 25–33.

The author provides a constructive proof of the minmax theorem of von Neumann which is elementary in that it uses purely algebraic methods. An adaptation of the simplex method for solving the linear programming problem provides directly an optimal mixed strategy for each player and the value of the game. The method is quite efficient as there are very few steps in addition to usual steps of the simplex method. An example is worked out. *E. D. Nering* (Tucson, Ariz.).

Blyth, Conrad Alexander. The theory of capital and its time measures. Econometrica 24 (1956), 467–479.

This is an ingenious attempt to measure the time dimensions of capital. The well-known difficulties in the concept of period of production are avoided by introducing additional time measures. Let $x(j)$ be the output at time j, r the instantaneous rate of interest, and $X(r)$ the present value of the outputs discounted at rate r. Let $M_x(-r)$ be the moment-generating function derived

by considering $x(j)$ as a frequency function. Then,

$$X(r)=X(0)\cdot M_x(-r).$$

Thus the revenue stream is determined by the moments of the output distribution over time; the period of production is the first moment. Similar considerations apply to the input stream.

The maximization conditions for optimal choice of production pattern can be expressed in terms of the cumulants of the input and output streams. Some other applications are given, including formulas for the value of capital based on different assumptions. *K. J. Arrow.*

Sargan, John D. A note on Mr. Blyth's article. Econometrica 24 (1956), 480–481.

The author suggests another set of time measures of capital which start from the streams of discounted (instead of undiscounted) inputs and outputs, as in the paper reviewed above. *K. J. Arrow* (Stanford, Calif.).

Gorra, Pierre. Une méthode d'élaboration de nombres-indices permanents. Metroecon. 5 (1953), 31–34.

By using the index numbers of Laspeyres and Paasche, only those goods can be taken into account which are present in the market at both points of time to be compared. The author suggests, that this principle should be used also for the quantities, by taking into account only the quantities of goods on the market at both these time points. On this ground the author suggests that instead of the formulas of Laspeyres and Paasche the formula

$$I_p=\frac{S(p_t q_e)}{S(p_0 q_e)}$$

where $q_e=\min(q_t,\ q_0)$ should be used. The suggested quantity index is

$$I_p=\frac{S(p_t q_t)}{I_p\cdot S(p_0 q_0)}.$$

{The author's suggestion is peculiar in the sense that if all quantities $q_t>q_0$ the Laspeyres formula should be used but if all $q_t<q_0$ then Paasche formula should be used, even if there are no other reasons for the fulfilment of the relations required than an increased, respectively decreased, number of buyers in the market. If corrections are made for this obvious arbitrariness the simplicity of the method is lost. Even if the method would be improved by changing q_e to $\min(q_0 I_q,\ q_t)$, I cannot see that the method would have any theoretical or practical merits compared with other formulas.} *L. Törnqvist.*

Debreu, Gerard. Market equilibrium. Proc. Nat. Acad. Sci. U.S.A. 42 (1956), 876–878.

Let S be the unit sphere in the Euclidean space R^l of l dimensions, C a closed convex cone in the same space with 0 as vertex, Γ its polar, $\zeta(p)$ an upper semicontinuous and bounded multivalued function which is homogeneous of degree 0 from $C\cap S$ to R^l and such that for every p in $C\cap S$ the set $\zeta(p)$ is nonempty, convex, and satisfies the condition $p\cdot z\leqq 0$ for every z in $\zeta(p)$. Then for some p in $C\cap S$, $\Gamma\cap\zeta(p)$ is nonempty. The theorem arises from the theory of economic equilibrium; p is a price vector and $\zeta(p)$ the excess demands compatible with p. The very simple proof is partially motivated by economic considerations. *K. J. Arrow* (Stanford, Calif.).

Fei, John Ching-Han. A fundamental theorem for the aggregation problem of input-output analysis. Econometrica 24 (1956), 400–412.

Let A be a Leontief matrix, A^* a Leontief matrix in which several sectors have been aggregated into one, and $A^*{}_*$ a matrix obtained from A^* by repeating the coefficients for the aggregated sector in the rows and columns of each of its subsectors. Then it is shown that $(I-A^*)^{-1}$ can also be obtained from $(I-A^*{}_*)^{-1}$ by aggregation, and that $(I-A^*{}_*)^{-1}=I-I_*+(I-A^*)_*{}^{-1}$. This theorem is related to the estimation of errors in Leontief inverses. *H. S. Houthakker* (Stanford, Calif.).

Dantzig, George B. Note on Klein's "Direct use of extremal principles in solving certain problems involving inequalities." Operations Res. 4 (1956), 247–249.

Certain classes of variational problems can occasionally be simplified by replacing inequalities of the form $f_i(x_1, x_2, \cdots, x_m)\leqq 0$ by equalities of the form

$$f_i(x_1, x_2, \cdots, x_m)+u_i^2=0.$$

This method, which has been used extensively in the past, in particular by Valentine in the calculus of variations [Contributions to the calculus of variations 1933–1937, Univ. of Chicago Press, 1937, pp. 403–447], was recently proposed by B. Klein in connection with the numerical solution of linear programming problems [J. Operations Res. Soc. Amer. 3 (1955), 168–175, 548; MR 16, 937]. The author analyzes this method in detail, shows that it is equivalent to usual search over vertices, and demonstrates that it is, for problems of any magnitude, far inferior to the simplex method. *R. Bellman.*

Koopman, B. O. The theory of search. I. Kinematic bases. Operations Res. 4 (1956), 324–346.

First of a series of three expository papers on methods used during the war to attack problems of submarine search, etc. *J. Kiefer* (Ithaca, N.Y.).

★**Churchman, C. West; Ackoff, Russell L.; and Arnoff, E. Leonard. Introduction to operations research.** John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London, 1957. x+645 pp. $12.00.

A handbook-textbook presupposing no more than elementary calculus. Techniques are presented without proof. Numerous examples and case studies. *J. Isbell.*

Prager, William. Management science: a new field for applied mathematicians. Pi Mu Epsilon J. 2 (1956), 204–219.

Fennell, Joseph; and Oshiro, Seiki. The dynamics of overhaul and replenishment systems for large equipments. Naval Res. Logist. Quart. 3 (1956), 19–43.

A Markov chain model is formulated for equipment which is subject to regular overhaul and in addition may have to be scrapped or to be subject to emergency overhaul on a random basis. A number of implications are drawn; in particular, the input schedule required to maintain a given time pattern of expected operational strength is found. Some further extensions are given. *K. J. Arrow* (Stanford, Calif.).

Bellman, Richard. Dynamic programming and the smoothing problem. Management Sci. 3 (1956), 111–113.

Generalizing results by A. J. Hoffman, W. Jacobs, and the reviewer [Management Sci. 1 (1954), 86–91, 92–95;

MR 17, 507] on the minimization of linear forms, the author gives a computational solution, by the methods of dynamic programming, of the problem of minimizing $\sum_1^n \phi_i(x_i)$ subject to the constraints $x_i \geqq 0$, $\sum_1^k x_i \geqq r_k$, where each $\phi_i(x)$ is monotone increasing for $x \geqq 0$ and $\{r_k\}$ is a given sequence of constants. *H. A. Antosiewicz.*

Verhulst, Michel. The concept of a "mission". Naval Res. Logist. Quart. 3 (1956), 45–57.

Discussion of a type of maximizing problem which may be called a "mission". The paper concludes with a theorem in multistage maximization. The reviewer cannot follow the simplifying assumptions at the beginning of the proof. *J. Isbell* (Princeton, N.J.).

Ward, J. B. Principles of programming. Elec. Engrg. 75 (1956), 1078–1083.

An expository account of a hypothetical computer, schematically presented with some examples of simple programming.

Castañeda, Jose. Linear programming and economic theory. Applications of linear programming. Trabajos Estadist. 7 (1956), 97–122. (Spanish)

Chenery, Hollis B.; and Kretschmer, Kenneth S. Resource allocation for economic development. Econometrica 24 (1956), 365–399.

The paper considers the problem of finding an optimal investment and development program for an economic system planning for a fixed target date. The scarce resources to be economized are labor, investment capacity, and foreign exchange. The technology is assumed to be of the input-output type, with the exception that capital per unit of output in each sector is taken as an increasing linear function of the fraction of output met from domestic production. In addition it is assumed that the system faces a falling linear demand for its exports (whose sole function is to buy imports). These two non-linearities turn the problem into one of non-linear programming in which total investment in domestic and export capacity is minimized subject to the availability of labor and foreign exchanged, an to the technology of domestic industries. The input-output nature of the technology allows an iterative solution for the shadow-prices or dual variables or Lagrange multipliers. Initial values P^w and P^f, the shadow-prices for labor and foreign exchange, are selected and a natural iterative process leads to the unique corresponding set of shadow-prices for domestically-produced commodities. The corresponding outputs, imports and exports are then computed and checked against the labor and foreign exchange constraints. P^w and P^f are then adjusted in the proper (supply-demand) directions and the process repeated until a complete solution is found. An example is worked out for the case of Italy; convergence is rapid. *R. Solow.*

See also: Hoffman and Kuhn, p. 370.

Biology and Sociology

Huber, Hans. Über die Anwendung statistischer Trennfunktionen zur Unterscheidung nahe verwandter Arten. Verh. Naturf. Ges. Basel 67 (1956), 149–175.

The paper discusses the application of discriminant function analyses to the biological problems of distinguishing between closely related species. Using only a single characteristics may be difficult or impossible, but a suitably weighted linear function of measurements on several characters may give adequate discrimination. Apart from standard methods the author also gives a sequential procedure for carrying out a discriminant analysis. {So far as the reviewer can discover this seems to be new in the form given, as well as relatively simple. The problem of sequential discrimination has been discussed by C. L. Mallows [Sankhyā 12 (1953), 321–338; MR 15, 453], but it is not immediately obvious how this is related to Huber's suggestion. The latter is probably quite good though perhaps not of maximum efficiency.} *N. T. J. Bailey* (Oxford).

Macey, Robert. A probabilistic approach to some problems in blood-tissue exchange. Bull. Math. Biophys. 18 (1956), 205–217.

Blood, entering a series of tissues arranged in parallel, contains a substance X whose concentration is suddenly changed from C_0 to C. In the ith tissue, of volume V_i and through which the rate of blood flow is r_i, the concentration of X is x_i, and that in blood leaving the tissue is $y_i = \alpha x_i$. Let $\lambda_i = r_i V_i$, $V(\lambda_i)$ be the probability density of λ_i, and

$$L(\alpha t) = \int_0^\infty V(\lambda) \exp(-\lambda \alpha t)\, d\lambda.$$

Then

$$L(\alpha t) = [\phi(\infty) - \phi(t)]\alpha/(C - C_0),$$

where $\phi(t)$ is the amount of X taken up by the tissue, by time t. The moments of $V(\lambda)$ are obtained as usual by differentiating $L(\alpha t)$ with respect to t. These results are generalised for the cases where (a) M tissue regions are arranged in parallel, each with a characteristic value of α and $V(\lambda)$; (b) X is consumed at a rate proportional to x, with a decay factor k, and $W(k, \lambda)$ is the bivariate distribution of k and λ (particular attention is paid to the case where k and λ are independent variables); (c) there is a bivariate distribution of λ and a permeability coefficient, h. *P. Armitage* (London).

Dusi, Teresa. La rappresentanza proporzionale. Una rappresentazione geometrica. Period. Mat. (4) 34 (1956), 220–227.

Rashevsky, N. Studies in mathematical biosociology of imitative behavior. I. Effects of income distribution. Bull. Math. Biophys. 18 (1956), 323–336.

Lev, Joseph. Maximizing test battery prediction when the weights are required to be non-negative. Psychometrika 21 (1956), 245–252.

See also: Anscombe, p. 425; Gilbert, p. 426.

Information and Communication Theory

Maximon, L. C.; and Ruina, J. P. Some statistical properties of signal plus narrow band noise integrated over a finite time interval. J. Appl. Phys. 27 (1956), 1442–1448.

See also: Trucco, p. 407; Thompson, p. 416.

Control Systems

Kavanagh, R. J. The application of matrix methods to multi-variable control systems. J. Franklin Inst. **262** (1956), 349–367.

The elementary formal side of the theory of linear control systems with several inputs and several outputs is neatly presented in terms of matrix algebra. There is no discussion of questions of stability, physical realizability or accuracy. *L. A. MacColl.*

Aĭzerman, M. A.; and Gantmaher, F. R. On the determination of periodic regimes in a non-linear dynamic system with piece-wise linear characteristic. Prikl. Mat. Meh. **20** (1956), 639–654. (Russian)

The system under discussion is

$$\dot{x}=\alpha x+\rho f(x_1),$$

where x and ρ are n-vectors, α is a constant square matrix and f is a scalar function. It is assumed that f is piece-wise linear (several pieces) and the calculations are carried out in the obvious and only possible manner. There are 12 references to automatic regulation.

S. Lefschetz (Mexico D.F.).

See also: Korobkov, p. 372; Troickiĭ, p. 395.

HISTORY, BIOGRAPHY

Heller, Siegfried. Ein Beitrag zur Deutung der Theodoros-Stelle in Platons Dialog "Theaetet". Centaurus **5** (1956), 1–58.

In this famous passage Plato tells us that Theodorus proved the irrationality of $\sqrt{3}$, $\sqrt{5}$ etc. up through $\sqrt{17}$. The puzzle has always been: why, after so many special cases, was he still unable to give a general proof. The explanation given in the present article is based on analogy with the geometric proof for $\sqrt{2}$ rather than, as in the attempts of other scholars, on the usual arithmetic proof. Details of the Greek text are discussed in an enlightening and convincing way. *S. H. Gould* (Providence, R.I.).

Hofmann, Jos. E. Ergänzende Bemerkungen zum "geometrischen" Irrationalitätsbeweis der alten Griechen. Centaurus **5** (1956), 59–72.

Three supplementary remarks on certain details of the preceding article.

Thorndike, Lynn. Notes upon some medieval Latin astronomical, astrological and mathematical manuscripts at the Vatican. Isis **47** (1956), 391–404.

This article belongs to a series of similar articles by the same author, one of which was reviewed in MR **10**, 667.

Vetter, Q. Ein Fund von Zahlzeichen aus dem Bronzezeitalter. Wiss. Z. Karl-Marx-Univ. Leipzig. Math.-Nat. Reihe **5** (1955/56), 131–132.

Biermann, Kurt-R. Aus der Geschichte der Wahrscheinlichkeitsrechnung. Wiss. Ann. **5** (1956), 542–548.

Müller, Conrad. Descartes' "Geometrie" und die Begründung der höheren Analysis. Sudhoffs Arch. **40** (1956), 240–258.

Picone, Mauro. Vito Volterra. Ricerca Sci. **26** (1956), 3277–3289.

A eulogy (with one photograph) of Volterra (1860–1940) under the headings: scienziato, interprete del progresso scientifico, senatore del regno d'Italia, la società italiana per il progresso delle scienze, le ricerche talassografiche, prima guerra mondiale e consiglio nazionale delle ricerche italiano, accademia dei Lincei.

Born, M. Erinnerungen an Albert Einstein. Math. Naturwiss. Unterricht **9** (1956/57), 97–105.

Campedelli, Luigi. Federigo Enriques nella scienza e nella scuola. Archimede **8** (1956), 97–103.

A eulogy of the scientific and pedagogical career of Enriques (1871–1946), with three photographs.

Behnke, Heinrich. Wilhelm Lorey zum Gedächtnis. Math.-Phys. Semesterber. **5** (1956), 1–3 (1 plate).

★ **Carathéodory, Constantin. Gesammelte mathematische Schriften.** Bd. 4. Herausgegeben im Auftrag und mit Unterstützung der bayerischen Akademie der Wissenschaften. C. H. Beck'sche Verlagsbuchhandlung, München, 1956. ix+494 pp.+1 plate. DM 43.

The fourth volume contains 7 articles on functions of a complex variable, 5 on functions of several complex variables, and 12 on real variables. The articles on complex variables are a continuation of the third volume [MR 17, 446], and those on real variables are entitled "Masstheorie", "eineindeutige Abbildung", and "Algebraisierung des Integralbegriffs".

See also: Noguera, p. 381; Bianchi, p. 413.

MISCELLANEOUS

★ **Newman, James R. The world of mathematics.** Vols. I–IV. A small library of the literature of mathematics from A'h-mosé the Scribe to Albert Einstein, presented with commentaries and notes. Simon & Schuster, New York, 1956. xviii+2535 pp. $20.00 (four volumes)

This anthology of mathematical essays by mathematicians and other writers has already been reviewed in many journals. Here it is only necessary to say that these essays will be as interesting to mathematical readers as to others. Many of them are examples of the art (more difficult in mathematics than in any other science?) of "haute vulgarisation", and the editor's commentaries are a fitting accompaniment. Some of the subjects dealt with are: biography of mathematicians, counting, space and motion, the physical world, social science, chance, statistics, design of experiments, group theory, logic, mathematical machines, warfare, art, ethics, literature and music. The writers include Archimedes, Boole, Descartes, Eddington, Galileo, Halley, Hardy, Jeans, Keynes, Laplace, Mendel, Newton, Poincaré, Russell, Shaw, Whitehead and many others. *S. H. Gould* (Providence, R.I.).

Schillo, Paul. A mathematical Munchausen. Math. Mag. 30 (1956), 55–61.

★ **Rademacher, Hans; and Toeplitz, Otto. The enjoyment of mathematics; Selections from mathematics for the amateur.** Princeton University Press, Princeton, N. J., 1957. 204 pp. $4.50.

This is a translation by Herbert Zuckerman from Von Zahlen und Figuren: Proben Mathematischen Denkens für Liebhaber der Mathematik, 2nd edit., Springer, Berlin, 1933. Chapters 15 and 28 by Herbert Zuckerman have been added to the English language edition.

★ **Smith, Arthur. The game of Go, the national game of Japan.** Charles E. Tuttle Co., Rutland, Vermont & Tokyo, Japan, 1956. (Originally published, 1908 by Moffat, Yard & Co., New York. Photographic copy.) xv+224 pp. $1.75.

The game of Go has a great fascination for mathematicians because the simplicity of its rules make them seem like a system of axioms for a conventional mathematical system with many beautiful theorems. The present book is a highly recommendable practical guide for the Westerner.

★ **v. Mangoldt, H. Einführung in die höhere Mathematik. Für Studierende und zum Selbststudium. Seit der fünften Auflage neu herausgegeben und erweitert von Konrad Knopp. Erster Band. Zahlen, Funktionen, Grenzwerte, analytische Geometrie, Algebra, Mengenlehre. 10., vollständig neubearbeitete Auflage.** S. Hirzel Verlag, Stuttgart, 1956. xvi+564 pp. DM 23.00.

The ninth edition was reviewed in MR 11, 87. In the "Vorwort zur 10. Auflage" the author states: "Die Gesamtanlage des Werkes und die Stoffauswahl im Grossen ist ··· die gleiche geblieben. Aber die immer wiederholte Durcharbeitung des ganzen Stoffes in Vorlesungen und Übungen zusammen mit den Ratschlägen von aussen haben es wünschenswert gemacht, die innere Gestaltung zu erneuern und die Darstellungen allenthalben weiter zu klären und zu bessern — so durchgreifend, dass es nicht möglich ist, in Kürze die Änderungen im einzelnen zu bezeichnen."

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